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RECONCILING AUSTINIAN AND RUSSELLIAN ACCOUNTS OF THE LIAR PARADOX

INTRODUCING *THE LIAR*

Barwise and Etchemendy's *The Liar* presents two accounts of the semantics of the liar sentence "this sentence is not true", named for Russell and Austin, and informed by situation semantics and non-well-founded set theory. The accounts differ over how a speaker uses a sentence and the world to express a proposition.

> By a *statement* we will understand certain sorts of datable events, those where a speaker asserts or attempts to assert something using a declarative sentence. In contrast, we take a *proposition* to be a claim about the world, the kind of thing that is asserted by a successful statement. (*The Liar*, page 11)

Under the Russellian account,

> sentences are used to express propositions, claims about the world, and these claims are true just in case the world is as it is claimed to be. (*The Liar*, page 26)

Thus, a Russellian statement is an utterance of a sentence, the proposition expressed by that statement is a claim about the world, and the proposition is true iff the claim holds. Under the Austinian account,

> a legitimate statement A provides two things: a historical (or actual) situation s_A, and a type of situation T_A. The former is just some limited portion of the real world; the speaker refers to it with what Austin [in his paper "Truth"] calls "demonstrative conventions." The latter is, roughly speaking, a property of situations

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> determined from the statement by means of "descriptive conventions" associated with the language. The statement A is true if s_A is of type T_A; otherwise it is false.
>
> ...
>
> While Austin did not use the term "proposition," it seems in the spirit of his account to identify what we will call the *Austinian proposition* expressed by A with the claim that s_A is of type T_A, and to individuate such a proposition by its two components, the situation referred to and the type of situation it is claimed to be. (*The Liar*, pages 28 and 29)

Thus, an Austinian statement is a reference to a situation and an utterance of a sentence, the proposition expressed by that statement is a claim that the situation is of a type, and that proposition is true iff the claim holds.

The Liar models both accounts using standard model theory built on non-standard set theory.

> Since the semantic phenomena that concern us involve circularity of various sorts, standard set theories, all of which assume the axiom of foundation, are quite awkward in that they foreclose the most natural ways of modeling these phenomena. Because of this, we have turned to an elegant alternative due to Peter Aczel. In this theory the axiom of foundation is replaced with an "anti-foundation" axiom, called AFA. This axiom is based on an extremely intuitive alternative to the cumulative conception of sets, and guarantees the existence of a rich class of circular objects with which to model the circular phenomena involved. (*The Liar*, pages 18 and 19)

Under the Russellian account, the liar occurs in only one statement, and so it expresses only one proposition. *The Liar* shows that this proposition is untrue. Therefore, it cannot be a fact in the world that the liar proposition is true. However, *The Liar* also shows that it cannot be a fact in the world that the liar proposition is untrue.

> What are we to make of this odd consequence of the Russellian account? If we take it seriously, it does indeed yield a diagnosis of the paradox, but a rather unsettling one. From this perspective, where our intuitive reasoning goes wrong is in thinking that the world encompasses everything that is the case. Give up this assumption, and the paradox is avoided: the Liar is not true, but this fact cannot be a fact in the world, a fact that can be truly described. But this is a rather big assumption to give up. (*The Liar*, page 105)

Rather than infer that the world lacks factual integrity, *The Liar* concludes that Russellian propositions cannot make claims about the whole world.

> That is, we can think of the Russellian's world as simply *part* of the real total world, a part that a proposition can be about. Of course, it cannot encompass everything there is, and so there remain facts . . . which lie beyond the scope of the Russellian. (*The Liar*, pages 155 and 156)

Under the Austinian account, the liar occurs in statements that refer to different situations, and so it expresses a family of propositions indexed by the situations. *The Liar* shows that each of these propositions is untrue. Therefore, again it cannot be a fact in the world that a liar proposition is true. However, the Austinian account does not generate factual 'gaps'. *The Liar* shows that it is a fact in the world that the proposition expressed by a liar statement is untrue. The only caveat is that this fact cannot be in the situation the statement refers to.

> In our Russellian development, we were confronted with an unintuitive partiality of the world. Liar-like propositions generated a host of second-class "facts" concerning their truth values which could not actually

> be incorporated into the world, on pain of paradox. This partiality does not infect the Austinian world: the truth value of every proposition is a first-class fact, a genuine constituent of the world. Yet there remains an essential partiality. The partiality is not a property of the world itself, but of those parts of the world that propositions can be about. Or if we think of it in terms of language, we see that while the world is as total as one could want, we cannot, in general, make statements about the world as a whole. (*The Liar*, page 154)

Though *The Liar* concludes that Russellian propositions express claims about a proper part of the world, it could be that the part of the world is not limited enough to be an Austinian situation. *The Liar* develops the Reflection Theorem (*The Liar*, theorem 20, page 157) to show that even so, the Austinian account subsumes the Russellian. The theorem shows that there is a situation s such that the Russellian expression of the statement comprising an utterance of a sentence σ is true iff the Austinian expression of the statement comprising a reference to s and an utterance of σ is true.

> Thus, a Russellian can always think of himself as expressing a proposition about the whole world, but an Austinian will view it as being about some ... situation (*The Liar*, page 156)

However, the Reflection Theorem only shows that the Austinian account subsumes part of the Russellian account. The Russellian propositions form a proper class, whereas the Russellian statements form a set. Thus, there are Russellian propositions that are not expressions of Russellian statements, yet the Reflection Theorem shows only that expressions of Russellian statements have truth equivalent Austinian counterparts.

INTRODUCING THIS PAPER

This paper unearths a compelling symmetry between the Russellian and Austinian accounts, and exploits this symmetry to construct an embedding that fully immerses the Russellian account in the Austinian. It identifies a Löwenheim–Skolem component of the Reflection Theorem, and isolates this component as a new theorem. Finally, it combines the embedding and theorem to yield a pellucid new proof of an expanded Reflection Theorem.

Studying the liar requires a syntax capable of generating the sentences uttered in statements that make assertions concerning propositions referred to by that statement. *The Liar* uses one syntax for both accounts. A term indicates an object, and a proposition is an object. A predicate indicates a relation among objects and an atomic sentence comprises a predicate and a sequence of terms. A compound sentence comprises a negation, conjunction or disjunction of simpler sentences. This paper uses a modest generalisation of *The Liar*'s syntax.

Since propositions cannot be directly manipulated, *The Liar* constructs sets to function as models of propositions. This paper uses a variant of *The Liar*'s definition of a model of an Austinian proposition, but uses a variant of an alternative definition of a model of a Russellian proposition due to Peter Aczel.[1] Both of this paper's definitions of a model of a proposition employ the situation semantics notion of an infon,[2] and yield models isomorphic to the models *The Liar*'s original definitions yield.

This paper models a Russellian proposition as follows.

1. Model an infon with a tuple comprising a relation, some objects and a polarity. Propositions are objects. $\langle \pi, x_1, \ldots, x_n, i \rangle$ models the infon that relation π holds (if $i = 1$) or does not hold (if $i = 0$) of the objects $x_1, \ldots, x_n$.
2. Model a proposition with an infon, or a conjunction or disjunction of simpler propositions. Infon i models the proposition that i is the case, conjunction $\langle \wedge, x \rangle$ models the proposition that each proposition in x is the case, and disjunction $\langle \vee, x \rangle$ models the proposition that some proposition in x is the case.

There is an obvious circularity in this definition. Models of propositions are defined in terms of models of infons, which are defined in terms of models of propositions! This paper escapes the circularity by choosing the largest classes of models that obey steps 1 and 2.

This paper models an Austinian proposition as follows.

1. Model an infon with a tuple comprising a relation, some objects and a polarity. Propositions are objects. $\langle\pi, x_1, \ldots, x_n, i\rangle$ models the infon that relation π holds (if $i = 1$) or does not hold (if $i = 0$) of the objects $x_1, \ldots, x_n$.
2. Model a situation with a set of infons. Each and only each infon in s is the case in the situation s models.
3. Model a type with an infon, or a conjunction or disjunction of simpler types. Infon i models the type of situation in which i is the case, conjunction $\langle\wedge, x\rangle$ models the type of situation that is of each type in x, and disjunction $\langle\vee, x\rangle$ models the type of situation that is of some type in x.
4. Model a proposition with a pair comprising a situation and a type. $\langle s, t\rangle$ models the proposition that situation s is of type t.

There is an obvious circularity in this definition, too. This paper escapes the circularity by again choosing the largest classes of models that obey steps 1–4.

Some propositions are true, and *The Liar* has methods to identify those sets that model true propositions. This paper adapts *The Liar*'s methods to suit this paper's models.

The Liar posits

> a basic relation that holds between a set of facts and a Russellian proposition just in case the facts make the proposition true. Then a Russellian proposition will be true just in case there is a set of facts that makes it true, and false just in case there is no such set. (*The Liar*, page 27)

Thus, a Russellian proposition is true iff some set of facts from the world makes it true. *The Liar* calls a set of facts a Russellian situation. To identify those sets that model true Russellian propositions this paper must model a Russellian situation, define the makes true relation and model the world.

This paper models a Russellian situation with a set of infons. Each and only each infon in s is a fact in the situation s models. If i is an infonic proposition then situation s makes i true if $i \in s$. If $\langle\wedge, x\rangle$ is a conjunctive proposition then situation s makes $\langle\wedge, x\rangle$ true iff s makes p true for each $p \in x$. If $\langle\vee, x\rangle$ is a disjunctive proposition then situation s makes $\langle\vee, x\rangle$ true iff s makes p true for some $p \in x$. This paper models a world with a class of Russellian infons. Each and only each member of W is the case in the world W models. If W is a model of the world then Russellian proposition p is true iff some set $s \subseteq W$ makes p true.

Identifying those sets that model true Austinian propositions needs no model of the world. If i is an infonic type then proposition $\langle s, i\rangle$ is true iff $i \in s$. If $\langle\wedge, x\rangle$ is a conjunctive type then proposition $\langle s, \langle\wedge, x\rangle\rangle$ is true iff $\langle s, t\rangle$ is true for each $t \in x$. If $\langle\vee, x\rangle$ is a disjunctive type then proposition $\langle s, \langle\vee, x\rangle\rangle$ is true iff $\langle s, t\rangle$ is true for some $t \in x$.

Successful statements express propositions. A statement made using an atomic sentence expresses the proposition that the relation indicated by the predicate holds of the objects indicated by the terms. A statement made using the negation of sentence σ expresses the proposition that the expression of a statement made using σ is not the case. A statement made using the conjunction (disjunction) of sentences σ_1 and σ_2 expresses the proposition that the expression of a statement made using σ_1 holds and (or) the expression of a statement made using σ_2 holds.

This seems straightforward enough, but expressing a statement can pose formidable problems. *The Liar*'s syntax includes a term **this**. Suppose **this** occurs in atomic sentence σ and Σ is a statement made using σ. Each occurrence of **this** in σ indicates the proposition expressed by Σ. Thus, the expression of Σ has the expression of Σ as a component!

Despite this, *The Liar* defines a function that assigns each Russellian statement a model of the proposition it expresses, and a function that assigns each Austinian statement a model of the proposition it expresses. This paper also defines an expression function for each account, but these definitions differ radically from *The Liar*'s. For each account however, this paper's expression function and *The Liar*'s expression function assign a statement models of the same proposition.

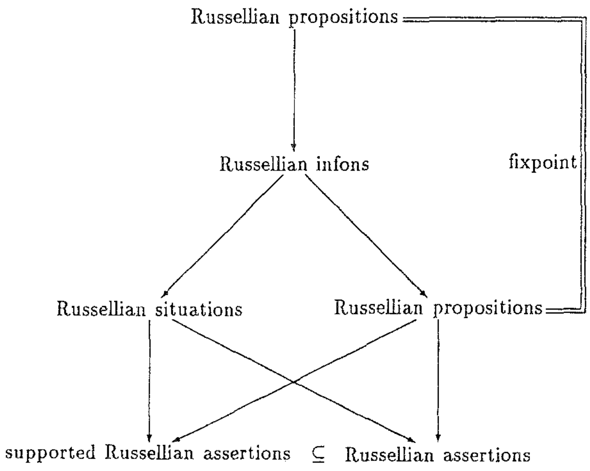
Notice an arresting symmetry between the way certain Russellian models relate to models of Russellian propositions and the way certain Austinian models relate to models of Austinian propositions:

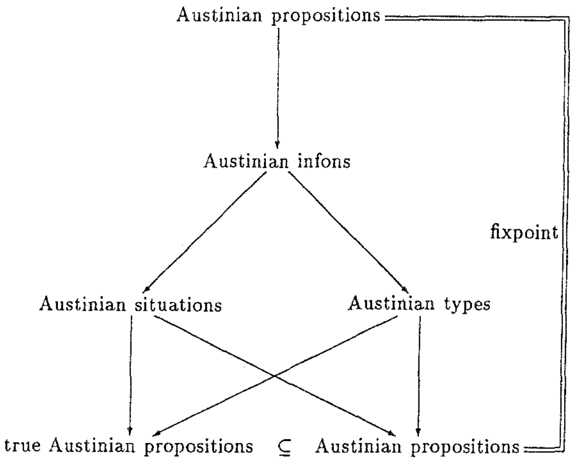
Russellian propositions generate Russellian infons, situations and propositions in exactly the way that Austinian propositions generate Austinian infons, situations and types.

Neither Aczel nor *The Liar* has a Russellian equivalent of an Austinian proposition, but this paper creates one. It defines a Russellian assertion as a claim that a situation makes a proposition true, and models a Russellian assertion with a pair comprising a situation and a proposition. $\langle s, p\rangle$ models the assertion that situation s makes proposition p true. This paper calls Russellian assertion $\langle s, p\rangle$ supported iff s makes p true. The symmetry of the previous paragraph continues: Russellian propositions generate Russellian assertions and supported assertions in exactly the way that Austinian propositions generate Austinian propositions and true propositions. This symmetry is depicted on the following page.

This paper exploits this symmetry to define an injection $\mathfrak{Emb}$ from the Russellian assertions to the Austinian propositions, and gives a proof (due jointly to Peter Aczel and the author) that if a is a Russellian assertion then a is supported iff $\mathfrak{Emb}(a)$ is true. Consequently, if p is a Russellian proposition then for some Russellian situation s, p is true iff $\mathfrak{Emb}\langle s, p\rangle$ is true. This paper then uses $\mathfrak{Emb}$ to define a function $\{\!|\ |\!\}$ from the Russellian situations to the Austinian situations, and proves that if s is a Russellian situation, σ is a sentence and p is the Russellian expression of a statement comprising an utterance of σ then $\mathfrak{Emb}\langle s, p\rangle$ is the Austinian expression of a statement comprising a reference to $\{\!|s|\!\}$ and an utterance of σ. Thus, each Russellian proposition r has an Austinian counterpart a such that r is true iff a is true, and if r is the Russellian expression of a statement made using sentence σ then a is the Austinian expression of a statement also made using σ. Consequently, the Austinian account subsumes all of the Russellian account.

$\mathfrak{Emb}$ does not yield the Reflection Theorem. Like the theorem, $\mathfrak{Emb}$ gives each sentence σ an Austinian situation s such that the Russellian expression of an utterance of σ is true iff the Austinian expression of a reference to s and an utterance of σ is true. But unlike the theorem, $\mathfrak{Emb}$ does not give each sentence the same situation. This is because the Reflection Theorem is "a kind of





Löwenheim–Skolem Theorem" (*The Liar*, exercise 64, page 160). It replaces a large world with a small situation, yet preserves the veracity of expressed statements. This paper presents a new Löwenheim–Skolem theorem, one that stays within the Russellian account. This Refraction Theorem shows that there is a Russellian situation s such that the Russellian expression p of a statement is true iff s makes p true. This paper shows that $\mathfrak{Emb}$ and the Refraction Theorem together give a new, perspicuous proof of an expanded

Reflection Theorem that avoids the artifices and exotica (such as protected situations) of *The Liar*'s original proof.

THE ACZEL LEGACY

Neither *The Liar* nor this paper would exist without Peter Aczel's work on non-well-founded sets. This section outlines the pertinent parts of Aczel's work. Aczel's *Non-Well-Founded Sets* contains the proofs this section omits.

Any set s can be pictured by a rooted and directed graph in which the nodes of the graph are labelled with sets such that s labels the root, and each node is labelled by the set of its children's labels. However, well-founded set theory forbids that all graphs picture a set, since a graph containing a cycle pictures no well-founded set. Aczel's insight is to ignore this injunction and posit that any graph can picture a set. If $\mathfrak{X}$ is a function defined on the nodes of a directed graph such that $\mathfrak{X}(x) = \{\mathfrak{X}(y) | y \text{ is a child of } x\}$ then call $\mathfrak{X}$ a decoration of that graph.

AXIOM 1. THE ANTI-FOUNDATION AXIOM. *Each directed graph has a unique decoration.*

Both *The Liar* and this paper work in Zermelo–Fraenkel set theory with atoms and the axiom of choice, but with the foundation axiom replaced by the anti-foundation axiom. Let *Atom* be the class of atoms, *Sets* be the class of sets and *Univ* be the universe of sets and atoms.

If $\mathfrak{X}$ is a function on a class X of atoms and $\mathfrak{Y}$ is a function on the universe such that

$$\mathfrak{Y}(x) = \begin{cases} \mathfrak{X}(x) & \text{if } x \in X \\ x & \text{if } x \in \mathit{Atom} \backslash X \\ \{\mathfrak{Y}(y) | y \in x\} & \text{if } x \in \mathit{Sets}. \end{cases}$$

then call $\mathfrak{Y}$ a substitution of $\mathfrak{X}$.

LEMMA 2. THE SUBSTITUTION LEMMA. *Each function on a class of atoms has a unique substitution.*

Write $\tilde{\mathfrak{X}}$ for the unique substitution of $\mathfrak{X}$.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are functions on a class of atoms such that $\mathfrak{Y}(x) = \tilde{\mathfrak{Y}}(\mathfrak{X}(x))$ then call $\mathfrak{Y}$ a solution of $\mathfrak{X}$. If $\mathfrak{X}$ is a function on a class of atoms such that $Range(\mathfrak{X}) \subseteq Sets$ then call $\mathfrak{X}$ a system of equations.

LEMMA 3. THE SOLUTION LEMMA. *Each system of equations has a unique solution.*

A class operator $\mathfrak{X}$ assigns each class X a unique class $\mathfrak{X}(X)$. Call class operator $\mathfrak{X}$ monotone iff $\mathfrak{X}(X) \subseteq \mathfrak{X}(Y)$ whenever $X \subseteq Y$, and call $\mathfrak{X}$ set based iff $y \in \mathfrak{X}(x)$ for some set $x \subseteq X$ whenever $y \in \mathfrak{X}(X)$. Call a class operator set continuous iff it is monotone and set based. Call class X a fixpoint of class operator $\mathfrak{X}$ iff $X = \mathfrak{X}(X)$. Call fixpoint X of class operator $\mathfrak{X}$ a largest fixpoint of $\mathfrak{X}$ iff $Y \subseteq X$ whenever $Y \subseteq \mathfrak{X}(Y)$.

LEMMA 4. THE LARGEST FIXPOINT LEMMA. *Each set continuous class operator has a unique largest fixpoint.*

LEMMA 5. *If $\mathfrak{Y}$ and $\mathfrak{Z}$ are set continuous class operators then so is $\mathfrak{X}$, where $\mathfrak{X}(X) = \mathfrak{Z}(\mathfrak{Y}(X))$ for each class X.*

Proof. $\mathfrak{X}$ is easily monotone. To show that $\mathfrak{X}$ is set based, let $z \in \mathfrak{X}(X)$. There must be set $y \subseteq \mathfrak{Y}(X)$ such that $z \in \mathfrak{Z}(y)$. $t \in \mathfrak{Y}(X)$ for each $t \in y$, so there must be set $x_t \subseteq X$ such that $t \in \mathfrak{Y}(x_t)$. Let $x = \cup\{x_t | t \in y\}$. Then $y \subseteq \mathfrak{Y}(x)$. Hence, $z \in \mathfrak{X}(x)$. □

LEMMA 6. *If $\mathfrak{Y}$ and $\mathfrak{Z}$ are set continuous class operators then so is $\mathfrak{X}$, where $\mathfrak{X}(X) = \mathfrak{Y}(X) \times \mathfrak{Z}(X)$ for each class X.*

Proof. $\mathfrak{X}$ is easily monotone. To show that $\mathfrak{X}$ is set based, let $\langle u, v\rangle \in \mathfrak{X}(X)$. There must be sets $y, z \subseteq X$ such that $u \in \mathfrak{Y}(y)$ and $v \in \mathfrak{Z}(z)$. Let $x = y \cup z$. Then $\langle u, v\rangle \in \mathfrak{X}(x)$. □

SENTENCES

Each term indicates an object. Each term is either a name, a demonstrative or a sentence. Each name indicates an object other than a proposition, each demonstrative indicates a proposition and each sentence indicates the proposition it is used to express. Each term has a sort according to the kind of object it indicates. Let *prop* be a distinguished sort. Each term of sort *prop* indicates a proposition. Thus, each demonstrative and sentence is of sort *prop*, but each name is of some other sort.

Each predicate indicates a relation among objects. Each predicate has an arity according to the sort of terms it takes to form an atomic sentence. If the arity of predicate π is $\langle s_1, \ldots, s_n \rangle$, the sort of term τ_1 is $s_1, \ldots$, the sort of term τ_n is s_n then $(\pi\tau_1 \ldots \tau_n)$ is an atomic sentence. Let **true** be a distinguished predicate with arity $\langle prop \rangle$. **true** indicates than an object is true. Each compound sentence is constructed from simpler sentences by negation, conjunction, disjunction or the marking of a special scope.

Let *Name* be the set of names, *Pred* be the set of predicates and *Sort* be the set of sorts. Let $\mathfrak{Sort}'$ be the function that assigns each name its sort and $\mathfrak{Arit}$ be the function that assigns each predicate its arity. Names, predicates, sorts, name sort assignments and predicate arity assignments vary for each language. Let *Demo* be the set of demonstratives. Demonstratives are fixed for each language. $Demo = \{\textbf{this}, \textbf{that}_1, \textbf{that}_2, \textbf{that}_3, \ldots\}$. Let *Term* be the set of terms and *Sent* be the set of sentences. Let $\mathfrak{Sort}$ be the function that assigns each term its sort. Terms, sentences and term sort assignments are defined simultaneously. *Term* and *Sent* are the smallest sets such that[3]

if $\nu \in Name$ then $\nu \in Term$ and $\mathfrak{Sort}(\nu) = \mathfrak{Sort}'(\nu)$,
if $\delta \in Demo$ then $\delta \in Term$ and $\mathfrak{Sort}(\delta) = prop$,
if $\sigma \in Sent$ then $\sigma \in Term$ and $\mathfrak{Sort}(\sigma) = prop$,
if $\pi \in Pred$, $\tau_1 \in Term, \ldots, \tau_n \in Term$
  and $\mathfrak{Arit}(\pi) = \langle \mathfrak{Sort}(\tau_1), \ldots, \mathfrak{Sort}(\tau_n) \rangle$
  then $(\pi\tau_1 \ldots \tau_n) \in Sent$,
if $\sigma \in Sent$ then **not** $\sigma \in Sent$,
if $\sigma_1 \in Sent$ and $\sigma_2 \in Sent$ then $(\sigma_1$ **and** $\sigma_2) \in Sent$,
if $\sigma_1 \in Sent$ and $\sigma_2 \in Sent$ then $(\sigma_1$ **or** $\sigma_2) \in Sent$, and
if $\sigma \in Sent$ then $\downarrow \sigma \in Sent$.

PROPOSITIONS

Let each name and predicate be an atom. Let 0, 1, $\wedge$ and $\vee$ be atoms also. Assume *Name*, *Pred* and $\{0, 1, \wedge, \vee\}$ are pairwise disjoint. For each class X, let $\mathfrak{Sort}_X : Name \cup X \to Sort$ by

$$\mathfrak{Sort}_X(u) = \begin{cases} \mathfrak{Sort}'(u) & \text{if } u \in Name \\ prop & \text{otherwise;} \end{cases}$$

let

$$\mathfrak{Info}(X) = \{\langle \pi, x_1, \ldots, x_n, i \rangle \in Pred \times (Name \cup X)^* \times \times \{0, 1\} | \mathfrak{Arit}(\pi) = \langle \mathfrak{Sort}_X(x_1), \ldots, \mathfrak{Sort}_X(x_n) \rangle\};$$

let

$$\mathfrak{Situ}(X) = \{x \in Sets | x \subseteq X\};$$

and let $\mathfrak{Type}(X)$ be the smallest class Y such that
if $x \in X$ then $x \in Y$,
if set $x \subseteq Y$ then $\langle \wedge, x \rangle \in Y$, and
if set $x \subseteq Y$ then $\langle \vee, x \rangle \in Y$.
Clearly, $\mathfrak{Info}$ and $\mathfrak{Situ}$ are set continuous class operators.

PROPOSITION 7. *$\mathfrak{Type}$ is a set continuous class operator.*

Proof. $\mathfrak{Type}$ is easily monotone. For each class X, let

$$\mathfrak{X}(X) = \cup\{\mathfrak{Type}(x) | \text{set } x \subseteq X\}.$$

To show that $\mathfrak{Type}$ is set based, it suffices to show that $\mathfrak{Type}(X) \subseteq \mathfrak{X}(X)$ for each class X.

$$\begin{aligned} & x \in X \\ & \Rightarrow \{x\} \subseteq X \text{ and } x \in \mathfrak{Type}(\{x\}) \\ & \Rightarrow x \in \mathfrak{X}(X). \end{aligned} \quad \text{(i)}$$

For each $y \in \mathfrak{X}(X)$, choose set $a_y \subseteq X$ such that $y \in \mathfrak{Type}(a_y)$. For each set $x \subseteq \mathfrak{X}(X)$, let $b_x = \cup\{a_y | y \in x\}$.

$$\begin{aligned} & x \subseteq \mathfrak{X}(X) \\ & \Rightarrow x \subseteq \mathfrak{Type}(b_x) \\ & \Rightarrow \langle \wedge, x \rangle \in \mathfrak{Type}(b_x) \text{ and } \langle \vee, x \rangle \in \mathfrak{Type}(b_x) \\ & \Rightarrow \langle \wedge, x \rangle \in \mathfrak{X}(X) \text{ and } \langle \vee, x \rangle \in \mathfrak{X}(X). \end{aligned} \quad \text{(ii)}$$

By (i) and (ii), $\mathfrak{Type}(X) \subseteq \mathfrak{X}(X)$. □

For each class X, let

$$\mathfrak{Russ}(X) = \mathfrak{Type}(\mathfrak{Info}(X)),$$

and let

$$\mathfrak{Aust}(X) = \mathfrak{Situ}(\mathfrak{Info}(X)) \times \mathfrak{Type}(\mathfrak{Info}(X)).$$

Both $\mathfrak{Russ}$ and $\mathfrak{Aust}$ are set continuous class operators by Lemmata 5 and 6, and so both have unique largest fixpoints by Lemma 4. Let $Prop_R$ be the largest fixpoint of $\mathfrak{Russ}$ and let $Prop_A$ be the largest fixpoint of $\mathfrak{Aust}$.

Let

$$Info_R = \mathfrak{Info}(Prop_R),$$
$$Situ_R = \mathfrak{Situ}(Info_R), \text{ and}$$
$$Asst_R = Situ_R \times Prop_R.$$

Notice that

$$Prop_R = \mathfrak{Type}(Info_R).$$

The members of $Prop_R$, $Info_R$, $Situ_R$ and $Asst_R$ are models of Russellian propositions, infons, situations and assertions respectively.[4]

Let

$$Info_A = \mathfrak{Info}(Prop_A),$$
$$Situ_A = \mathfrak{Situ}(Info_A), \text{ and}$$
$$Type_A = \mathfrak{Type}(Info_A).$$

Notice that

$$Prop_A = Situ_A \times Type_A.$$

The members of $Prop_A$, $Info_A$, $Situ_A$ and $Type_A$ are models of Austinian propositions, infons, situations and types respectively.[5]

TRUE PROPOSITIONS

For each class X, let $\mathfrak{Supp}(X)$ be the smallest class Y such that
if $x \in \mathfrak{Situ}(X)$ and $y \in x$ then $\langle x, y \rangle \in Y$,

if $x \in \mathfrak{Situ}(X)$, set $y \subseteq \mathfrak{Type}(X)$ and $\langle x, z\rangle \in Y$ for each $z \in y$
then $\langle x, \langle \wedge, y\rangle\rangle \in Y$, and
if $x \in \mathfrak{Situ}(X)$, set $y \subseteq \mathfrak{Type}(X)$ and $\langle x, z\rangle \in Y$ for some $z \in y$
then $\langle x, \langle \vee, y\rangle\rangle \in Y$.

Let

$$Supp_R = \mathfrak{Supp}(Info_R)$$

and

$$Supp_A = \mathfrak{Supp}(Info_A).$$

If X is a class then

$$\mathfrak{Supp}(X) \subseteq \mathfrak{Situ}(X) \times \mathfrak{Type}(X).$$

Thus,

$$Supp_R \subseteq Asst_R$$

and

$$Supp_A \subseteq Prop_A.$$

The members of $Supp_R$ and $Supp_A$ model supported Russellian assertions and true Austinian propositions respectively.

$Supp_R$ does not model Russellian truth. A Russellian proposition is true iff the world is as the proposition claims it is. Therefore, a fully modelled Russellian semantics needs a model of the real world. On the other hand, $Supp_A$ does model Austinian truth. However, not every member of $Situ_A$ models a real situation, a limited part of the real world. Therefore, a fully modelled Austinian semantics also needs a model of the real world. Each class of Russellian infons is a model of a Russellian world, and each class of Austinian infons is a model of an Austinian world. Each member of W is a fact in the world W models.

If W is a class of Russellian infons and $p \in Prop_R$ then let $W \models p$ iff $\langle s, p\rangle \in Supp_R$ for some set $s \subseteq W$. Thus, $W \models p$ iff some situation in the world W models makes the proposition p models true. If W models the real world then $W \models p$ iff p models a true proposition. Thus, $\models$ models Russellian truth.

Of course, one class of infons cannot be isolated as a model of the real world, Russellian or Austinian. However, certain classes of infons can be rejected as models of the real world. Call a world incoherent iff some infon and its dual are facts in that world. Call a world dishonest iff the truth of some proposition is a fact in that world yet the proposition is untrue, or the untruth of some proposition is a fact in that world yet the proposition is true. Call class W of Russellian infons weak iff

if $\langle \pi, x_1, \ldots, x_n, i\rangle \in W$ then $\langle \pi, x_1, \ldots, x_n, 1-i\rangle \notin W$,
if $\langle \mathbf{true}, p, 1\rangle \in W$ then $W \models p$, and
if $\langle \mathbf{true}, p, 0\rangle \in W$ then $W \not\models p$.

Similarly, call class W of Austinian infons weak[6] iff

if $\langle \pi, x_1, \ldots x_n, i\rangle \in W$ then $\langle \pi, x_1, \ldots, x_n, 1-i\rangle \notin W$,
if $\langle \mathbf{true}, p, 1\rangle \in W$ then $p \in Supp_A$, and
if $\langle \mathbf{true}, p, 0\rangle \in W$ then $p \notin Supp_A$.

A model of a coherent and honest world must be a weak class of infons. Intuitively, the real world is both coherent and honest. Therefore, if our intuitions are correct then a model of the real world must be a weak class of infons.

EXPRESSION

Let

$$Cont_R = \{\mathfrak{C} | \mathfrak{C} : \{\mathbf{that}_1, \mathbf{that}_2, \mathbf{that}_3, \ldots\} \to Prop_R \cup Sent\}$$

and

$$Cont_A = \{\mathfrak{C} | \mathfrak{C} : \{\mathbf{that}_1, \mathbf{that}_2, \mathbf{that}_3, \ldots\} \to Prop_A \cup Sent\}.$$

the members of $Cont_R$ and $Cont_A$ are Russellian and Austinian contexts respectively. A context gives the proposition indicated by each occurrence of a **that** demonstrative. If $\mathfrak{C}(\mathbf{that}_i)$ is a proposition then each occurrence of $\mathbf{that}_i$ indicates $\mathfrak{C}(\mathbf{that}_i)$. If $\mathfrak{C}(\mathbf{that}_i)$ is a sentence then each occurrence of $\mathbf{that}_i$ indicates the proposition expressed by a statement made using $\mathfrak{C}(\mathbf{that}_i)$.[7]

Unfortunately, giving the proposition indicated by each occurrence of the **this** demonstrative is less straightforward. Essentially, each occurrence of **this** indicates the proposition expressed by a statement made using the sentence in which it occurs. However, the first occurrence of **this** in

(**not**(**true this**) **and** (**true this**))

for example, occurs in three sentences:

(**true this**),

not(**true this**), and

(**not**(**true this**) **and** (**true this**)).

It could feasibly indicate the proposition expressed by a statement made using any of these three.

The scope symbol ↓ disambiguates this ambivalence. An occurrence o of **this** in a sentence s indicates the proposition expressed by a statement made using the largest subsentence s' of s containing o such that s' does not begin with ↓ and each subsentence of s' containing o does not begin with ↓. For example, the first **this** in

(↓ **not**(**true this**) **and** (**true this**))

indicates the proposition expressed by a statement made using

(**not** (**true this**)),

and the second **this** indicates the proposition expressed by a statement made using

(↓ **not**(**true this**) **and** (**true this**)).

Let $* : \{0, 1\} \times \{\textbf{and}, \textbf{or}\} \to \{\wedge, \vee\}$ by

$1 * \textbf{and} = \wedge$,

$1 * \textbf{or} = \vee$,

$0 * \textbf{and} = \vee$ and

$0 * \textbf{or} = \wedge$.

If $\mathfrak{C} \in Cont_R$ then let $\mathfrak{Val}_R^{\mathfrak{C}} : Sent \times Sent \times \{0, 1\} \to Prop_R$ by

$$\mathfrak{Val}_R^{\mathfrak{C}}\langle(\pi\tau_1 \ldots \tau_n), \rho, i\rangle = \langle\pi, x_1, \ldots, x_n, i\rangle$$

where $x_j =$

$$= \begin{cases} \tau_j & \text{if } \tau_j \in \mathit{Name} \\ \mathfrak{Val}^{\mathfrak{C}}_R\langle \tau_j, \rho, 1\rangle & \text{if } \tau_j \in \mathit{Sent} \\ \mathfrak{Val}^{\mathfrak{C}}_R\langle \rho, \rho, 1\rangle & \text{if } \tau_j \text{ is } \textbf{this} \\ \mathfrak{C}(\tau_j) & \text{if } \tau_j \text{ is } \textbf{that}_k \\ & \text{and } \mathfrak{C}(\tau_j) \in \mathit{Prop}_R \\ \mathfrak{Val}^{\mathfrak{C}}_R\langle \mathfrak{C}(\tau_j), \mathfrak{C}(\tau_j), 1\rangle & \text{if } \tau_j \text{ is } \textbf{that}_k \text{ and } \mathfrak{C}(\tau_j) \in \mathit{Sent}, \end{cases}$$

$$\mathfrak{Val}^{\mathfrak{C}}_R\langle \textbf{not } \sigma, \rho, i\rangle = \mathfrak{Val}^{\mathfrak{C}}_R\langle \sigma, \rho, 1-i\rangle,$$

$$\mathfrak{Val}^{\mathfrak{C}}_R\langle (\sigma_1 \textbf{ and } \sigma_2), \rho, i\rangle =$$

$$= \langle i * \textbf{ and}, \{\mathfrak{Val}^{\mathfrak{C}}_R\langle \sigma_1, \rho, i\rangle, \mathfrak{Val}^{\mathfrak{C}}_R\langle \sigma_2, \rho, i\rangle\}\rangle,$$

$$\mathfrak{Val}^{\mathfrak{C}}_R\langle (\sigma_1 \textbf{ or } \sigma_2), \rho, i\rangle =$$

$$= \langle i * \textbf{ or}, \{\mathfrak{Val}^{\mathfrak{C}}_R\langle \sigma_1, \rho, i\rangle, \mathfrak{Val}^{\mathfrak{C}}_R\langle \sigma_2, \rho, i\rangle\}\rangle, \text{ and}$$

$$\mathfrak{Val}^{\mathfrak{C}}_R\langle \downarrow \sigma, \rho, i\rangle = \mathfrak{Val}^{\mathfrak{C}}_R\langle \sigma, \sigma, i\rangle.$$

$\mathfrak{Val}^{\mathfrak{C}}_R\langle \sigma, \rho, i\rangle$ is almost the Russellian expression of a statement made using sentence σ, except that any unscoped occurrence of **this** in σ indicates the proposition expressed by a statement made using ρ, and that the expression is negated if $i = 0$.

Since the definition of $\mathfrak{Val}^{\mathfrak{C}}_R$ is non-well-founded, it is not immediately clear that $\mathfrak{Val}^{\mathfrak{C}}_R$ is well-defined. The proof of the following proposition uses the solution lemma (Lemma 3) to show that $\mathfrak{Val}^{\mathfrak{C}}_R$ is well-defined. It is typical of many of the proofs in this paper.

PROPOSITION 8. *If* $\mathfrak{C} \in \mathit{Cont}_R$ *then* $\mathfrak{Val}^{\mathfrak{C}}_R$ *is well-defined.*

Proof. If $\sigma \in \mathit{Sent}$, $\rho \in \mathit{Sent}$ and $i \in \{0, 1\}$ then let $[\sigma, \rho, i] \in \mathit{Atom}$. Let $\mathfrak{X}$ be the system of equations

$$\mathfrak{X}[(\pi\tau_1 \ldots \tau_n), \rho, i] = \langle \pi, x_1, \ldots, x_n, i\rangle$$

where $x_j =$

$$= \begin{cases} \tau_j & \text{if } \tau_j \in \mathit{Name} \\ [\tau_j, \rho, 1] & \text{if } \tau_j \in \mathit{Sent} \\ [\rho, \rho, 1] & \text{if } \tau_j \text{ is } \mathbf{this} \\ \mathfrak{C}(\tau_j) & \text{if } \tau_j \text{ is } \mathbf{that}_k \text{ and } \mathfrak{C}(\tau_j) \in \mathit{Prop}_R \\ [\mathfrak{C}(\tau_j), \mathfrak{C}(\tau_j), 1] & \text{if } \tau_j\ \mathbf{that}_k \text{ and } \mathfrak{C}(\tau_j) \in \mathit{Sent}, \end{cases}$$

$$\mathfrak{X}[\mathbf{not}\ \sigma, \rho, i] = \mathfrak{X}[\sigma, \rho, 1 - i],$$

$$\mathfrak{X}[(\sigma_1\ \mathbf{and}\ \sigma_2), \rho, i] = \langle i * \ \mathbf{and}, \{[\sigma_1, \rho, i], [\sigma_2, \rho, i]\}\rangle,$$

$$\mathfrak{X}[(\sigma_1\ \mathbf{or}\ \sigma_2), \rho, i] = \langle i * \ \mathbf{or}, \{[\sigma_1, \rho, i], [\sigma_2, \rho, i]\}\rangle, \text{ and}$$

$$\mathfrak{X}[\downarrow \sigma, \rho, i] = \mathfrak{X}[\sigma, \sigma, i].$$

By Lemma 3, $\mathfrak{X}$ has a unique solution $\mathfrak{Y}$. For each $\sigma \in \mathit{Sent}$, $\rho \in \mathit{Sent}$ and $i \in \{0, 1\}$, put

$$\mathfrak{Val}_R^{\mathfrak{C}}\langle \sigma, \rho, i\rangle = \mathfrak{Y}[\sigma, \rho, i].$$

Then $\mathfrak{Val}_R^{\mathfrak{C}}$ is a function obeying the requisite definition. Suppose $\mathfrak{Val}'$ is also a function obeying the requisite definition. For each $\sigma \in \mathit{Sent}$, $\rho \in \mathit{Sent}$ and $i \in \{0, 1\}$, let

$$\mathfrak{Y}'[\sigma, \rho, i] = \mathfrak{Val}'\langle \sigma, \rho, i\rangle.$$

Then $\mathfrak{Y}'$ is also a solution of $\mathfrak{X}$. Thus,

$$\mathfrak{Y}' = \mathfrak{Y},$$

and so

$$\mathfrak{Val}' = \mathfrak{Val}_R^{\mathfrak{C}}.$$

Thus, $\mathfrak{Val}_R^{\mathfrak{C}}$ is the unique function obeying the requisite definition.

It only remains to show that $\mathfrak{Val}_R^{\mathfrak{C}}$ has the requisite range.

$$\mathit{Prop}_R \subseteq \mathfrak{Russ}(\mathit{Prop}_R \cup \mathit{Range}(\mathfrak{Val}_R^{\mathfrak{C}}))$$

and, by induction on the complexity of σ,

$$\mathfrak{Val}_R^{\mathfrak{C}}\langle \sigma, \rho, i\rangle \in \mathfrak{Russ}(\mathit{Prop}_R \cup \mathit{Range}(\mathfrak{Val}_R^{\mathfrak{C}}))$$

for each $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0,1\}$. Therefore,

$$Prop_R \cup Range(\mathfrak{Val}_R^{\mathfrak{C}}) \subseteq \mathfrak{Russ}(Prop_R \cup Range(\mathfrak{Val}_R^{\mathfrak{C}})).$$

So, since $Prop_R$ is the largest fixpoint of $\mathfrak{Russ}$,

$$Prop_R \cup Range(\mathfrak{Val}_R^{\mathfrak{C}}) \subseteq Prop_R.$$

Thus,

$$Range(\mathfrak{Val}_R^{\mathfrak{C}}) \subseteq Prop_R. \qquad \square$$

The Russellian expression of a statement made using sentence σ is won from $\mathfrak{Val}_R^{\mathfrak{C}}$ by ensuring that each unscoped occurrence of **this** in σ indicates the proposition expressed by a statement made using σ, and that the expression is not negated. If $\mathfrak{C} \in Cont_R$ then let $\mathfrak{Exp}_R^{\mathfrak{C}} : Sent \rightarrow Prop_R$ by

$$\mathfrak{Exp}_R^{\mathfrak{C}}(\sigma) = \mathfrak{Val}_R^{\mathfrak{C}}\langle \sigma, \sigma, 1\rangle.$$

Call $\mathfrak{Exp}_R^{\mathfrak{C}}$ the Russellian expression function (under $\mathfrak{C}$).[8]

Austinian expression is similar to Russellian expression, but the situation referred to by an Austinian statement occasions a more complex definition. If $\mathfrak{C} \in Cont_A$ then let $\mathfrak{Val}_A^{\mathfrak{C}} : Situ_A \times Sent \times Sent \times \{0,1\} \rightarrow Prop_A$ and $\widehat{\mathfrak{Val}}_A^{\mathfrak{C}} : Situ_A \times Sent \times Sent \times \{0,1\} \rightarrow Type_A$ by

$$\mathfrak{Val}_A^{\mathfrak{C}}\langle s, \sigma, \rho, i\rangle = \langle s, \widehat{\mathfrak{Val}}_A^{\mathfrak{C}}\langle s, \sigma, \rho, i\rangle\rangle,$$

$$\widehat{\mathfrak{Val}}_A^{\mathfrak{C}}\langle s, (\pi\tau_1 \ldots \tau_n), \rho, i\rangle = \langle \pi, x_1, \ldots, x_n, i\rangle$$

$$\text{where } x_j = \begin{cases} \tau_j & \text{if } \tau_j \in Name \\ \mathfrak{Val}_A^{\mathfrak{C}}\langle s, \tau_j, \rho, 1\rangle & \text{if } \tau_j \in Sent \\ \mathfrak{Val}_A^{\mathfrak{C}}\langle s, \rho, \rho, 1\rangle & \text{if } \tau_j \text{ is } \mathbf{this} \\ \mathfrak{C}(\tau_j) & \text{if } \tau_j \text{ is } \mathbf{that}_k \\ & \text{and } \mathfrak{C}(\tau_j) \in Prop_A \\ \mathfrak{Val}_A^{\mathfrak{C}}\langle s, \mathfrak{C}(\tau_j), \mathfrak{C}(\tau_j), 1\rangle & \text{if } \tau_j \text{ is } \mathbf{that}_k \\ & \text{and } \mathfrak{C}(\tau_j) \in Sent, \end{cases}$$

$$\widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, \textbf{ not } \sigma, \rho, i\rangle = \widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, \sigma, \rho, 1-i\rangle,$$

$$\widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, (\sigma_1 \textbf{ and } \sigma_2), \rho, i\rangle$$

$$= \langle i * \textbf{ and}, \{\widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, \sigma_1, \rho, i\rangle, \widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, \sigma_2, \rho, i\rangle\}\rangle,$$

$$\widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, (\sigma_1 \textbf{ or } \sigma_2), \rho, i\rangle$$

$$= \langle i * \textbf{ or}, \{\widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, \sigma_1, \rho, i\rangle, \widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, \sigma_2, \rho, i\rangle\}\rangle, \text{ and}$$

$$\widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, \downarrow \sigma, \rho, i\rangle = \widehat{\mathfrak{Val}}^{\mathfrak{C}}_A\langle s, \sigma, \sigma, i\rangle.$$

As in the Russellian case, $\mathfrak{Val}^{\mathfrak{C}}_A\langle s, \sigma, \rho, i\rangle$ is almost the Austinian expression of a statement made using situation s and sentence σ, except that each unscoped occurrence of **this** is σ indicates the expression of a statement made using s and ρ, and that the expression is negated if $i = 0$.

PROPOSITION 9. *If* $\mathfrak{C} \in Cont_A$ *then* $\mathfrak{Val}^{\mathfrak{C}}_A$ *and* $\widehat{\mathfrak{Val}}^{\mathfrak{C}}_A$ *are well-defined.*

Proof. If $s \in Situ_A$, $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$ then let $[s, \sigma, \rho, i] \in Atom$ and $[\![s, \sigma, \rho, i]\!] \in Atom$. Let $\mathfrak{X}$ be the system of equations

$$\mathfrak{X}[s, \sigma, \rho, i] = \langle s, [\![s, \sigma, \rho, i]\!]\rangle,$$

$$\mathfrak{X}[\![s, (\pi\tau_1 \ldots \tau_n), \rho, i]\!] = \langle \pi, x_1, \ldots, x_n, i\rangle$$

where $x_j =$

$$= \begin{cases} \tau_j & \text{if } \tau_j \in Name \\ [s, \tau_j, \rho, 1] & \text{if } \tau_j \in Sent \\ [s, \rho, \rho, 1] & \text{if } \tau_j \text{ is } \textbf{this} \\ \mathfrak{C}(\tau_j) & \text{if } \tau_j \text{ is } \textbf{that}_k \text{ and } \mathfrak{C}(\tau_j) \in Prop_A \\ [s, \mathfrak{C}(\tau_j), \mathfrak{C}(\tau_j), 1] & \text{if } \tau_j \text{ is } \textbf{that}_k \text{ and } \mathfrak{C}(\tau_j) \in Sent, \end{cases}$$

$$\mathfrak{X}[\![s, \textbf{ not } \sigma, \rho, i]\!] = \mathfrak{X}[\![s, \sigma, \rho, 1-i]\!],$$

$$\mathfrak{X}[\![s, (\sigma_1 \textbf{ and } \sigma_2), \rho, i]\!] = \langle i * \textbf{ and}, \{[\![s, \sigma_1, \rho, i]\!], [\![s, \sigma_2, \rho, i]\!]\}\rangle,$$

$$\mathfrak{X}[\![(\sigma_1 \textbf{ or } \sigma_2), \rho, i]\!] = \langle i * \textbf{ or}, \{[\![s, \sigma_1, \rho, i]\!], [\![s, \sigma_2, \rho, i]\!]\}\rangle \text{ and}$$

$$\mathfrak{X}[\![s, \downarrow \sigma, \rho, i]\!] = \mathfrak{X}[\![s, \sigma, \sigma, i]\!].$$

By Lemma 3, $\mathfrak{X}$ has a unique solution $\mathfrak{Y}$. For each $s \in Situ_A$, $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$, put

$$\mathfrak{Val}_A^{\mathfrak{C}}\langle s, \sigma, \rho, i\rangle = \mathfrak{Y}[s, \sigma, \rho, i]$$

and

$$\widehat{\mathfrak{Val}}_A^{\mathfrak{C}}\langle s, \sigma, \rho, i\rangle = \mathfrak{Y}[\![s, \sigma, \rho, i]\!].$$

It only remains to show that $\mathfrak{Val}_A^{\mathfrak{C}}$ and $\widehat{\mathfrak{Val}}_A^{\mathfrak{C}}$ have the requisite ranges.

$$Prop_A \subseteq \mathfrak{Aust}(Prop_A \cup Range(\mathfrak{Val}_A^{\mathfrak{C}}))$$

and, by induction on the complexity of σ,

$$\widehat{\mathfrak{Val}}_A^{\mathfrak{C}}\langle s, \sigma, \rho, i\rangle \in \mathfrak{Russ}(Prop_A \cup Range(\mathfrak{Val}_A^{\mathfrak{C}}))$$

for each $s \in Situ_A$, $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$. Therefore,

$$Prop_A \cup Range(\mathfrak{Val}_A^{\mathfrak{C}}) \subseteq \mathfrak{Aust}(Prop_A \cup Range(\mathfrak{Val}_A^{\mathfrak{C}}))$$

and

$$Range(\widehat{\mathfrak{Val}}_A^{\mathfrak{C}}) \subseteq \mathfrak{Russ}(Prop_A \cup Range(\mathfrak{Val}_A^{\mathfrak{C}})).$$

So, since $Prop_A$ is the largest fixpoint of $\mathfrak{Aust}$,

$$Prop_A \cup Range(\mathfrak{Val}_A^{\mathfrak{C}}) \subseteq Prop_A$$

and

$$Range(\widehat{\mathfrak{Val}}_A^{\mathfrak{C}}) \subseteq \mathfrak{Russ}(Prop_A).$$

Thus,

$$Range(\mathfrak{Val}_A^{\mathfrak{C}}) \subseteq Prop_A$$

and

$$Range(\widehat{\mathfrak{Val}}_A^{\mathfrak{C}}) \subseteq Type_A. \qquad \square$$

The Austinian expression of a statement made using situation s and sentence σ is won from $\mathfrak{Val}_A^{\mathfrak{C}}$ by ensuring that each unscoped occurrence of **this** in σ indicates the proposition expressed by a statement made using s and σ, and that the expression is not negated. If $\mathfrak{C} \in Cont_A$ then let $\mathfrak{Exp}_A^{\mathfrak{C}} : Situ \times Sent \to Prop_A$ by

$$\mathfrak{Exp}_A^{\mathfrak{C}}\langle s, \sigma\rangle = \mathfrak{Val}_A^{\mathfrak{C}}\langle s, \sigma, \sigma, 1\rangle.$$

Call $\mathfrak{Exp}_A^{\mathfrak{C}}$ the Austinian expression function (under $\mathfrak{C}$).[9]

THE EMBEDDING

Let $\mathfrak{Emb} : Asst_R \to Prop_A$ and $\widehat{\mathfrak{Emb}} : Asst_R \to Type_A$ by

$$\mathfrak{Emb}\langle s, p\rangle = \langle\{\widehat{\mathfrak{Emb}}\langle s, i\rangle | i \in s\}, \widehat{\mathfrak{Emb}}\langle s, p\rangle\rangle,$$

$$\widehat{\mathfrak{Emb}}\langle s, \langle \pi, x_1, \ldots, x_n, i\rangle\rangle = \langle \pi, y_1, \ldots, y_n, i\rangle$$

$$\text{where } y_j = \begin{cases} x_j & \text{if } x_j \in Name \\ \mathfrak{Emb}\langle s, x_j\rangle & \text{if } x_j \in Prop_R, \end{cases}$$

$$\widehat{\mathfrak{Emb}}\langle s, \langle \wedge, x\rangle\rangle = \langle \wedge, \{\widehat{\mathfrak{Emb}}\langle s, p\rangle | p \in x\}\rangle, \text{and}$$

$$\widehat{\mathfrak{Emb}}\langle s, \langle \vee, x\rangle\rangle = \langle \vee, \{\widehat{\mathfrak{Emb}}\langle s, p\rangle | p \in x\}\rangle.$$

PROPOSITION 10. $\mathfrak{Emb}$ *and* $\widehat{\mathfrak{Emb}}$ *are well-defined.*

Proof. If $a \in Asst_R$ then let $[a] \in Atom$ and $[\![a]\!] \in Atom$. Let $\mathfrak{X}$ be the system of equations

$$\mathfrak{X}[\langle s, p\rangle] = \langle\{[\![\langle s, i\rangle]\!] | i \in s\}, [\![\langle s, p\rangle]\!]\rangle,$$

$$\mathfrak{X}[\![\langle s, \langle \pi, x_1, \ldots, x_n, i\rangle\rangle]\!] = \langle \pi, y_1, \ldots, y_n, i\rangle$$

$$\text{where } y_j = \begin{cases} x_j & \text{if } x_j \in Name \\ [\langle s, x_j\rangle] & \text{if } x_j \in Prop_R, \end{cases}$$

$$\mathfrak{X}[\![\langle s, \langle \wedge, x\rangle\rangle]\!] = \langle \wedge, \{[\![\langle s, p\rangle]\!] | p \in x\rangle\}, \text{and}$$

$$\mathfrak{X}[\![\langle s, \langle \vee, x\rangle\rangle]\!] = \langle \vee, \{[\![\langle s, p\rangle]\!] | p \in x\rangle\}.$$

By Lemma 3, $\mathfrak{X}$ has a unique solution $\mathfrak{Y}$. For each $a \in Asst_R$, put

$$\mathfrak{Emb}(a) = \mathfrak{Y}[a]$$

and

$$\widehat{\mathfrak{Emb}}(a) = \mathfrak{Y}[\![a]\!].$$

To show that $\mathfrak{Emb}$ and $\widehat{\mathfrak{Emb}}$ have the required ranges, let

$$Prop = \{\mathfrak{Emb}(a) | a \in Asst_R\},$$

$$Info = \{\widehat{\mathfrak{Emb}}\langle s, i\rangle | s \in Situ_R \text{ and } i \in Info_R\},$$

$$Situ = \{\{\widehat{\mathfrak{Emb}}\langle s, i\rangle | i \in s\} | s \in Situ_R\}, \text{and}$$

$$Type = \{\widehat{\mathfrak{Emb}}\langle s, p\rangle | s \in Situ_R \text{ and } p \in Prop_R\}.$$

Firstly,

$$Info \subseteq \mathfrak{Info}(Prop). \tag{i}$$

Secondly,

$$Situ \subseteq \mathfrak{Situ}(Info). \tag{ii}$$

Thirdly, for each $s \in Situ_R$, let

$$X_s = \{p \in Prop_R | \widehat{\mathfrak{Emb}}\langle s, p\rangle \in \mathfrak{Type}(Info)\}.$$

Then, for each $s \in Situ_R$,

if $i \in Info_R$ then $i \in X_s$,

if set $x \subseteq X_s$ then $\langle \wedge, x\rangle \in X_s$, and

if set $x \subseteq X_s$ then $\langle \vee, x\rangle \in X_s$.

Thus, for each $s \in Situ_R$,

$$Prop_R \subseteq X_s.$$

Therefore,

$$Type \subseteq \mathfrak{Type}(Info). \tag{iii}$$

Finally,

$$Prop \subseteq Situ \times Type. \tag{iv}$$

By (i)–(iv),

$$Prop \subseteq \mathfrak{Aust}(Prop).$$

Hence,

$$Prop \subseteq Prop_A$$

and, by (i) and (iii),

$$Type \subseteq Type_A.$$

Therefore,

$$Range(\mathfrak{Emb}) \subseteq Prop_A$$

and

$$Range(\widehat{\mathfrak{Emb}}) \subseteq Type_A. \qquad \square$$

This paper constructs quasi-inverses $\mathfrak{Inv}$ and $\widehat{\mathfrak{Inv}}$ of $\mathfrak{Emb}$ and $\widehat{\mathfrak{Emb}}$. Let $\mathfrak{Inv} : Prop_A \to Asst_R$ and $\widehat{\mathfrak{Inv}} : Type_A \to Prop_R$ by

$$\mathfrak{Inv}\langle s, t\rangle = \langle \{\widehat{\mathfrak{Inv}}(i) | i \in s\}, \widehat{\mathfrak{Inv}}(t)\rangle,$$

$$\widehat{\mathfrak{Inv}}\langle \pi, x_1, \ldots, x_n, i\rangle = \langle \pi, y_1, \ldots, y_n, i\rangle$$

$$\text{where } y_j = \begin{cases} x_j & \text{if } x_j \in Name \\ \widehat{\mathfrak{Inv}}(t) & \text{if } x_j \text{ is } \langle s, t\rangle \in Prop_A, \end{cases}$$

$$\widehat{\mathfrak{Inv}}\langle \wedge, x\rangle = \langle \wedge, \{\widehat{\mathfrak{Inv}}(t) | t \in x\}\rangle, \text{and}$$

$$\widehat{\mathfrak{Inv}}\langle \vee, x\rangle = \langle \vee, \{\widehat{\mathfrak{Inv}}(t) | t \in x\}\rangle.$$

PROPOSITION 11. *$\mathfrak{Inv}$ and $\widehat{\mathfrak{Inv}}$ are well-defined.*

Proof. It suffices to show that $\widehat{\mathfrak{Inv}}$ is well-defined. If $t \in Type_A$ then let $[t] \in Atom$. Let $\mathfrak{X}$ be the system of equations

$$\mathfrak{X}[\langle \pi, x_1, \ldots, x_n, i\rangle] = \langle \pi, y_1, \ldots, y_n, i\rangle$$

$$\text{where } y_j = \begin{cases} x_j & \text{if } x_j \in Name \\ [t] & \text{if } x_j \text{ is } \langle s, t\rangle \in Prop_A, \end{cases}$$

$$\mathfrak{X}[\langle \wedge, x\rangle] = \langle \wedge, \{[t] | t \in x\}\rangle, \text{and}$$

$$\mathfrak{X}[\langle \vee, x\rangle] = \langle \vee, \{[t] | t \in x\}\rangle.$$

By Lemma 3, $\mathfrak{X}$ has a unique solution $\mathfrak{Y}$. Put

$$\widehat{\mathfrak{Inv}}(x) = \mathfrak{Y}[x].$$

To show that $\widehat{\mathfrak{Inv}}$ has the required range, let

$$Prop = \{\widehat{\mathfrak{Inv}}(t) | t \in Type_A\}, \text{ and}$$

$$Info = \{\widehat{\mathfrak{Inv}}(i) | i \in Info_A\}.$$

Firstly,

$$Info \subseteq \mathfrak{Info}(Prop). \quad \text{(i)}$$

Secondly, let

$$X = \{t \in Type_A | \widehat{\mathfrak{Inv}}(t) \in \mathfrak{Type}(Info)\}.$$

Then

$$\text{if } i \in Info_A \text{ then } i \in X,$$

$$\text{if set } x \subseteq X \text{ then } \langle \wedge, x \rangle \in X, \text{ and}$$

$$\text{if set } x \subseteq X \text{ then } \langle \vee, x \rangle \in X.$$

Thus,

$$Type_A \subseteq X.$$

Therefore,

$$Prop \subseteq \mathfrak{Type}(Info). \quad \text{(ii)}$$

By (i) and (ii),

$$Prop \subseteq \mathfrak{Russ}(Prop).$$

Hence,

$$Prop \subseteq Prop_R.$$

Therefore,

$$Range(\widehat{\mathfrak{Inv}}) \subseteq Prop_R. \quad \square$$

LEMMA 12. *If* $s \in Situ_R$ *and* $p \in Prop_R$ *then* $\widehat{\mathfrak{Inv}}(\widehat{\mathfrak{Emb}}\langle s, p \rangle) = p$.

Proof. If $a \in Asst_R$ then let $[a] \in Atom$. Let $\mathfrak{X}$ be the system of equations

$$\mathfrak{X}[\langle s, \langle \pi, x_1, \ldots, x_n, i\rangle\rangle] = \langle \pi, y_1, \ldots, y_n, i\rangle$$

$$\text{where } y_j = \begin{cases} x_j & \text{if } x_j \in Name \\ [\langle s, x_j\rangle] & \text{if } x_j \in Prop_R, \end{cases}$$

$$\mathfrak{X}[\langle s, \langle \wedge, x\rangle\rangle] = \langle \wedge, \{[\langle s, p\rangle] | p \in x\}\rangle, \text{and}$$

$$\mathfrak{X}[\langle s, \langle \vee, x\rangle\rangle] = \langle \vee, \{[\langle s, p\rangle] | p \in X\}\rangle.$$

For each $s \in Situ_R$ and $p \in Prop_R$, put

$$\mathfrak{Y}[\langle s, p\rangle] = \widehat{\mathfrak{Inv}}(\widehat{\mathfrak{Emb}}\langle s, p\rangle)$$

and

$$\mathfrak{Z}[\langle s, p\rangle] = p.$$

Both $\mathfrak{Y}$ and $\mathfrak{Z}$ are solutions of $\mathfrak{X}$. By Lemma 3, $\mathfrak{X}$ has a unique solution. Thus, for each $s \in Situ_R$ and $p \in Prop_R$,

$$\widehat{\mathfrak{Inv}}(\widehat{\mathfrak{Emb}}\langle s, p\rangle) = p.$$ □

LEMMA 13. *If* $a \in Asst_R$ *then* $\mathfrak{Inv}(\mathfrak{Emb}(a)) = a$.

PROPOSITION 14. $\mathfrak{Emb}$ *is injective.*

PROPOSITION 15. $a \in Supp_R$ *iff* $\mathfrak{Emb}(a) \in Supp_A$.[10]

Proof. Let

$$S_R = \{a \in Asst_R | \mathfrak{Emb}(a) \in Supp_A\}, \text{and}$$

$$S_A = \{p \in Prop_A | \mathfrak{Inv}(p) \in Supp_R\}.$$

It suffices to show that $Supp_R \subseteq S_R$ and, by Lemma 13, that $Supp_A \subseteq S_A$.

CLAIM. $Supp_R \subseteq S_R$.

Proof of claim. Firstly,

$$s \in Situ_R \text{ and } p \in s$$

$$\Rightarrow \{\widehat{\mathfrak{Emb}}\langle s, i\rangle | i \in s\} \in Situ_A \text{ and}$$

$$\widehat{\mathfrak{Emb}}\langle s, p\rangle \in \{\widehat{\mathfrak{Emb}}\langle s, i\rangle | i \in s\}$$

$$\Rightarrow \langle \{\widehat{\mathfrak{Emb}}\langle s, i\rangle | i \in s\}, \widehat{\mathfrak{Emb}}\langle s, p\rangle\rangle \in Supp_A$$

$$\Rightarrow \mathfrak{Emb}\langle s, p\rangle \in Supp_A$$

$$\Rightarrow \langle s, p\rangle \in S_R.$$

Secondly,

$$s \in \mathfrak{Situ}_R,\ \text{set } x \subseteq Prop_R \text{ and } \langle s, p\rangle \in S_R \text{ for each } p \in x$$

$$\Rightarrow \{\widehat{\mathfrak{Emb}}\langle s, i\rangle | i \in s\} \in Situ_A, \text{set} \{\widehat{\mathfrak{Emb}}\langle s, p\rangle | p \in x\} \subseteq Type_A$$

$$\text{and } \langle \{\widehat{\mathfrak{Emb}}\langle s, i\rangle | i \in s\}, \widehat{\mathfrak{Emb}}\langle s, p\rangle\rangle \in Supp_A$$

$$\text{for each } p \in x$$

$$\Rightarrow \langle \{\widehat{\mathfrak{Emb}}\langle s, i\rangle | i \in s\}, \langle \wedge, \{\mathfrak{Emb}\langle s, p\rangle | p \in x\}\rangle\rangle \in Supp_A$$

$$\Rightarrow \mathfrak{Emb}\langle s, \langle \wedge, x\rangle\rangle \in Supp_A$$

$$\Rightarrow \langle s, \langle \wedge, x\rangle\rangle \in S_R.$$

Similarly,

$$s \in Situ_R,\ \text{set } x \subseteq Prop_R \text{ and } \langle s, p\rangle \in S_R \text{ for some } p \in x$$

$$\Rightarrow \langle s, \langle \vee, x\rangle\rangle \in S_R.$$

Thus,

$$Supp_R \subseteq S_R.$$ □

CLAIM. $Supp_A \subseteq S_A$.

Proof of claim. Firstly,

$$s \in Situ_A \text{ and } t \in s$$

$$\Rightarrow \{\widehat{\mathfrak{Inv}}(i) | i \in s\} \in Situ_R \text{ and } \widehat{\mathfrak{Inv}}(t) \in \{\widehat{\mathfrak{Inv}}(i) | i \in s\}$$

$$\Rightarrow \langle \{\widehat{\mathfrak{Inv}}(i) | i \in s\}, \widehat{\mathfrak{Inv}}(t)\rangle \in Supp_R$$

$$\Rightarrow \mathfrak{Inv}\langle s, t\rangle \in Supp_R$$

$$\Rightarrow \langle s, t\rangle \in S_A.$$

Secondly,

$$s \in Situ_A,\ \text{set } x \subseteq Type_A \text{ and } \langle s, t\rangle \in S_A \text{ for each } t \in x$$
$$\Rightarrow \{\widehat{\mathfrak{Inv}}(i) | i \in s\} \in Situ_R, \text{set} \{\widehat{\mathfrak{Inv}}(t) | t \in x\} \subseteq Prop_R$$
$$\text{and } \langle \{\widehat{\mathfrak{Inv}}(i) | i \in s\}, \widehat{\mathfrak{Inv}}(t)\rangle \in Supp_R \text{ for each } t \in x$$
$$\Rightarrow \langle \{\widehat{\mathfrak{Inv}}(i) | i \in s\}, \langle \wedge, \{\widehat{\mathfrak{Inv}}(t) | t \in x\}\rangle\rangle \in Supp_R$$
$$\Rightarrow \mathfrak{Inv}\langle s, \langle \wedge, x\rangle\rangle \in Supp_R$$
$$\Rightarrow \langle s, \langle \wedge, x\rangle\rangle \in S_A.$$

Similarly,

$$s \in Situ_A,\ \text{set } x \subseteq Type_A \text{ and } \langle s, t\rangle \in S_A \text{ for some } t \in x$$
$$\Rightarrow \langle s, \langle \vee, x\rangle\rangle \in S_A.$$

Thus,

$$Supp_A \subseteq S_A. \qquad \square\square$$

Let $\{\!\!\{\,\}\!\!\}: Situ_R \to Situ_A$ by

$$\{\!\!\{s\}\!\!\} = \{\widehat{\mathfrak{Emb}}\langle s, i\rangle | i \in s\}.$$

Let $*: Situ_R \times Cont_R \to Cont_A$ such that

$$s * \mathfrak{C}(\mathbf{that}_i) = \begin{cases} \mathfrak{Emb}\langle s, \mathfrak{C}(\mathbf{that}_i)\rangle & \text{if } \mathfrak{C}(\mathbf{that}_i) \in Prop_R \\ \mathfrak{C}(\mathbf{that}_i) & \text{if } \mathfrak{C}(\mathbf{that}_i) \in Sent. \end{cases}$$

LEMMA 16. *If* $\mathfrak{C} \in Cont_R$, $s \in Situ_R$, $\sigma \in Sent$, $\rho \in Sent$ *and* $i \in \{0, 1\}$ *then*

$$\mathfrak{Emb}\langle s, \mathfrak{Val}_R^{\mathfrak{C}}\langle \sigma, \rho, i\rangle\rangle = \mathfrak{Val}_A^{s*\mathfrak{C}}\langle \{\!\!\{s\}\!\!\}, \sigma, \rho, i\rangle.$$

Proof. Fix $s \in Situ_R$ and $\mathfrak{C} \in Cont_R$. If $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$ then let $[\sigma, \rho, i] \in Atom$ and $[\![\sigma, \rho, i]\!] \in Atom$. Let $\mathfrak{X}$ be the system of equations

$$\mathfrak{X}[\sigma, \rho, i] = \langle \{\!\!\{s\}\!\!\}, [\![\sigma, \rho, i]\!]\rangle,$$
$$\mathfrak{X}[\![(\pi\tau_1 \ldots \tau_n), \rho, i]\!] = \langle \pi, x_1, \ldots, x_n, i\rangle$$

where $x_j =$

$$= \begin{cases} \tau_j & \text{if } \tau_j \in \mathit{Name} \\ [\tau_j, \rho, 1] & \text{if } \tau_j \in \mathit{Sent} \\ [\rho, \rho, 1] & \text{if } \tau_j \text{ is } \mathbf{this} \\ \mathfrak{Emb}\langle s, \mathfrak{C}(\tau_j)\rangle & \text{if } \tau_j \text{ is } \mathbf{that}_k \text{ and } \mathfrak{C}(\tau_j) \in \mathit{Prop}_R \\ [\mathfrak{C}(\tau_j), \mathfrak{C}(\tau_j), 1] & \text{if } \tau_j \text{ is } \mathbf{that}_k \text{ and } \mathfrak{C}(\tau_j) \in \mathit{Sent}, \end{cases}$$

$$\mathfrak{X}[\![\mathbf{not}\ \sigma, \rho, i]\!] = \mathfrak{X}[\![\sigma, \rho, 1 - i]\!],$$

$$\mathfrak{X}[\![(\sigma_1\ \mathbf{and}\ \sigma_2), \rho, i]\!] = \langle i *\ \mathbf{and}\ , \{[\![\sigma_1, \rho, i]\!], [\![\sigma_2, \rho, i]\!]\}\rangle,$$

$$\mathfrak{X}[\![(\sigma_1\ \mathbf{or}\ \sigma_2), \rho, i]\!] = \langle i *\ \mathbf{or}, \{[\![\sigma_1, \rho, i]\!], [\![\sigma_2, \rho, i]\!]\}\rangle, \text{ and}$$

$$\mathfrak{X}[\![\downarrow \sigma, \rho, i]\!] = \mathfrak{X}[\![\sigma, \sigma, i]\!].$$

For each $\sigma \in \mathit{Sent}$, $\rho \in \mathit{Sent}$ and $i \in \{0, 1\}$, put

$$\mathfrak{Y}[\sigma, \rho, i] = \mathfrak{Emb}\langle s, \mathfrak{Val}_R^{\mathfrak{C}}\langle \sigma, \rho, i\rangle\rangle$$

and

$$\mathfrak{Y}[\![\sigma, \rho, i]\!] = \widehat{\mathfrak{Emb}}\langle s, \mathfrak{Val}_R^{\mathfrak{C}}\langle \sigma, \rho, i\rangle\rangle.$$

By induction on the complexity of σ,

$$\mathfrak{Y}[\![\sigma, \rho, i]\!] = \tilde{\mathfrak{Y}}(\mathfrak{X}[\![\sigma, \rho, i]\!]).$$

Therefore, $\mathfrak{Y}$ is a solution of $\mathfrak{X}$. For each $\sigma \in \mathit{Sent}$, $\rho \in \mathit{Sent}$ and $i \in \{0, 1\}$, put

$$\mathfrak{Z}[\sigma, \rho, i] = \mathfrak{Val}_A^{s * \mathfrak{C}}\langle \{\!|s|\!\}, \sigma, \rho, i\rangle$$

and

$$\mathfrak{Z}[\![\sigma, \rho, i]\!] = \widehat{\mathfrak{Val}}_A^{s * \mathfrak{C}}\langle \{\!|s|\!\}, \sigma, \rho, i\rangle.$$

By induction on the complexity of σ again,

$$\mathfrak{Z}[\![\sigma, \rho, i]\!] = \tilde{\mathfrak{Z}}(\mathfrak{X}[\![\sigma, \rho, i]\!]).$$

Therefore, $\mathfrak{Z}$ is also a solution of $\mathfrak{X}$. By Lemma 3, $\mathfrak{X}$ has a unique solution. Thus, for each $\sigma \in \mathit{Sent}$, $\rho \in \mathit{Sent}$ and $i \in \{0, 1\}$,

$$\mathfrak{Emb}\langle s, \mathfrak{Val}_R^{\mathfrak{C}}\langle \sigma, \rho, i\rangle\rangle = \mathfrak{Val}_A^{s * \mathfrak{C}}\langle \{\!|s|\!\}, \sigma, \rho, i\rangle.$$

□

PROPOSITION 17. *If* $\mathfrak{C} \in Cont_R$, $s \in Situ_R$ *and* $\sigma \in Sent$ *then*

$$\mathfrak{Emb}\langle s, \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle = \mathfrak{Exp}_A^{s*\mathfrak{C}}\langle \{\!|s|\!\}, \sigma\rangle.$$

THE REFRACTION THEOREM

LEMMA 18. *If* $\langle s,p\rangle \in Supp_R$, $s' \in Situ_R$ *and* $s \subseteq s'$ *then* $\langle s',p\rangle \in Supp_R$.

Proof. Let *Supp* be the unique class of Russellian assertions such that $\langle s,p\rangle \in Supp$ iff

if $s' \in Situ_R$ and $s \subseteq s'$ then $\langle s',p\rangle \in Supp_R$.

It suffices to show that $Supp_R \subseteq Supp$. Firstly,

$$s \in Situ_R \text{ and } p \in s$$
$$\Rightarrow \text{if } s' \in Situ_R \text{ and } s \subseteq s' \text{ then } p \in s'$$
$$\Rightarrow \text{if } s' \in Situ_R \text{ and } s \subseteq s' \text{ then } \langle s',p\rangle \in Supp_R$$
$$\Rightarrow \langle s,p\rangle \in Supp.$$

Secondly,

$$s \in Situ_R, \text{set } x \subseteq Prop_R \text{ and } \langle s,p\rangle \in Supp \text{ for each } p \in x,$$
$$\Rightarrow \text{if } s' \in Situ_R \text{ and } s \in s'$$
$$\text{then } s' \in Situ_R, \text{set } x \subseteq Prop_R$$
$$\text{and } \langle s',p\rangle \in Supp_R \text{ for each } p \in x$$
$$\Rightarrow \text{if } s' \in Situ_R \text{ and } s \subseteq s' \text{ then } \langle s', \langle \wedge, x\rangle\rangle \in Supp_R$$
$$\Rightarrow \langle s, \langle \wedge, x\rangle\rangle \in Supp.$$

Similarly,

$$s \in Situ_R, \text{set } x \subseteq Prop_R \text{ and } \langle s,p\rangle \in Supp \text{ for some } p \in x$$
$$\Rightarrow \langle s, \langle \vee, x\rangle\rangle \in Supp.$$

Thus,

$$Supp_R \subseteq Supp.$$ □

If $\mathfrak{C} \in Cont_R$, W is a class of Russellian infons and $s \in Situ_R$ such that

$$W \models \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma) \text{ iff } \langle s, \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle \in Supp_R$$

for each $\sigma \in Sent$ then call s a $\mathfrak{C}$-window of W. [11]

THEOREM 19. THE REFRACTION THEOREM. *If* $\mathfrak{C} \in Cont_R$ *and* W *is a weak class of Russellian infons then there is some set* $s \subseteq W$ *such that* s *is a weak* $\mathfrak{C}$*-window of* W.

Proof. Let

$$S = \{\sigma \in Sent | W \models \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\}.$$

Choose a function $\mathfrak{X} : S \to Pow(W)$ such that

$$\langle \mathfrak{X}(\sigma), \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle \in Supp_R$$

for each $\sigma \in S$. Let $\mathfrak{Y} : Pow(W) \to Pow(Prop_R)$ by

$$\mathfrak{Y}(s) = \{p \in Prop_R | \langle \mathbf{true}, p, 1\rangle \in s\}.$$

Then, for each set $s \subseteq W$,

$$\begin{aligned} & p \in \mathfrak{Y}(s) \\ & \Rightarrow \langle \mathbf{true}, p, 1\rangle \in s \\ & \Rightarrow \langle \mathbf{true}, p, 1\rangle \in W \\ & \Rightarrow W \models p. && \text{since } W \text{ is weak} \end{aligned}$$

For each set $s \subseteq W$, choose a function $\mathfrak{Z}_s : \mathfrak{Y}(s) \to Pow(W)$ such that

$$\langle \mathfrak{Z}_s(p), p\rangle \in Supp_R$$

for each $p \in \mathfrak{Y}(s)$. Let

$$s_0 = \cup Range(\mathfrak{X}).$$

For each $i \in \mathbb{N}$, let

$$s_{i+1} = s_i \cup (\cup Range(\mathfrak{Z}_{s_i})).$$

Let

$$s = \cup\{s_i | i \in \mathbb{N}\}.$$

CLAIM. *s is a set and $s \subseteq W$.*

Proof of claim. s_0 is a set and $s_0 \subseteq W$. For each $i \in \mathbb{N}$, if s_i is a set and $s_i \subseteq W$ then s_{i+1} is a set and $s_{i+1} \subseteq W$. Therefore, s is a set and $s \subseteq W$. □

CLAIM. *s is weak.*

Proof of claim. Firstly,

$$\begin{aligned} & \langle \pi, x_1, \ldots, x_n, 1\rangle \in s \text{ and } \langle \pi, x_1, \ldots, x_n, 0\rangle \in s \\ & \Rightarrow \langle \pi, x_1, \ldots, x_n, 1\rangle \in W \text{ and } \langle \pi, x_1, \ldots, x_n, 0\rangle \in W \\ & \Rightarrow \text{contradiction.} && \text{since W is weak} \end{aligned}$$

Therefore,
 if $\langle \pi, x_1, \ldots, x_n, i\rangle \in s$ then $\langle \pi, x_1, \ldots, x_n, 1-i\rangle \notin s$.
Secondly,

$$\begin{aligned}&\langle \textbf{true}, p, 1\rangle \in s\\ &\Rightarrow \text{for some } i \in \mathbb{N}, \langle \textbf{true}, p, 1\rangle \in s_i\\ &\Rightarrow \text{for some } i \in \mathbb{N}, p \in \mathfrak{Y}(s_i) \text{ and } \langle \mathfrak{Z}_{s_i}(p), p\rangle \in Supp_R\\ &\Rightarrow \text{for some } i \in \mathbb{N}, \mathfrak{Z}_{s_i}(p) \subseteq s_{i+1} \text{ and } \langle \mathfrak{Z}_{s_i}(p), p\rangle \in Supp_R\\ &\Rightarrow \text{for some } i \in \mathbb{N}, \mathfrak{Z}_{s_i}(p) \subseteq s \text{ and } \langle \mathfrak{Z}_{s_i}(p), p\rangle \in Supp_R\\ &\Rightarrow s \models p.\end{aligned}$$

Finally,

$$\begin{aligned}&\langle \textbf{true}, p, 0\rangle \in s \text{ and } s \models p\\ &\Rightarrow \langle \textbf{true}, p, 0\rangle \in W \text{ and } W \models p\\ &\Rightarrow \text{contradiction.} \qquad \text{since W is weak}\end{aligned}$$

Therefore,
 if $\langle \textbf{true}, p, 0\rangle \in s$ is then $s \not\models p$. □

CLAIM. *s is a* $\mathfrak{C}$*-window of W.*

Proof of claim.

$$\begin{aligned}&W \models \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\\ &\Rightarrow \sigma \in S\\ &\Rightarrow \langle \mathfrak{X}(\sigma), \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle \in Supp_R\\ &\Rightarrow \text{for some } s' \subseteq s_0, \langle s', \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle \in Supp_R\\ &\Rightarrow \text{for some } s' \subseteq s, \langle s', \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle \in Supp_R\\ &\Rightarrow \langle s, \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle \in Supp_R. \qquad \text{by Lemma 18}\end{aligned}$$

Conversely, it is immediately obvious that
 if $\langle s, \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle \in Supp_R$ then $W \models \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)$. □□

THE REFLECTION THEOREM

LEMMA 20. *If* $s \in Situ_R$ *is weak then* $\{\!|s|\!\}$ *is weak.*

Proof. Firstly,

$$\begin{aligned}&\langle \pi, x_1, \ldots, x_n, 1\rangle \in \{\!|s|\!\} \text{ and } \langle \pi, x_1, \ldots, x_n, 0\rangle \in \{\!|s|\!\}\\ &\Rightarrow \text{for some } i_1, i_2 \in s,\end{aligned}$$

$$\langle \pi, x_1, \ldots, x_n, 1\rangle = \widehat{\mathfrak{Emb}}\langle s, i_1\rangle \text{ and}$$

$$\langle \pi, x_1, \ldots, x_n, 0\rangle = \widehat{\mathfrak{Emb}}\langle s, i_2\rangle$$

$$\Rightarrow \text{for some } i_1, i_2 \in s,$$

$$\widehat{\mathfrak{Inv}}\langle \pi, x_1, \ldots, x_n, 1\rangle = \widehat{\mathfrak{Inv}}(\widehat{\mathfrak{Emb}}\langle s, i_1\rangle) \text{ and}$$

$$\widehat{\mathfrak{Inv}}\langle \pi, x_1, \ldots, x_n, 0\rangle = \widehat{\mathfrak{Inv}}(\widehat{\mathfrak{Emb}}\langle s, i_2\rangle)$$

$$\Rightarrow \widehat{\mathfrak{Inv}}\langle \pi, x_1, \ldots, x_n, 1\rangle \in s \text{ and}$$

$$\widehat{\mathfrak{Inv}}\langle \pi, x_1, \ldots, x_n, 0\rangle \in s \qquad \text{by Lemma 12}$$

$$\Rightarrow \text{for some } y_1, \ldots, y_n \in \mathit{Name} \cup \mathit{Prop}_R,$$

$$\langle \pi, y_1, \ldots, y_n, 1\rangle \in s \text{ and } \langle \pi, y_1, \ldots, y_n, 0\rangle \in s$$

$$\Rightarrow \text{contradiction.} \qquad \text{since } s \text{ is weak}$$

Therefore,

if $\langle \pi, x_1, \ldots, x_n, i\rangle \in \{\!|s|\!\}$ then $\langle \pi, x_1, \ldots, x_n, 1 - i\rangle \notin \{\!|s|\!\}$.

Secondly,

$$\langle \mathbf{true}, p, 1\rangle \in \{\!|s|\!\}$$

$$\Rightarrow \text{for some } i \in s, \widehat{\mathfrak{Emb}}\langle s, i\rangle = \langle \mathbf{true}, p, 1\rangle$$

$$\Rightarrow \text{for some } p' \in \mathit{Prop}_R, \langle \mathbf{true}, p', 1\rangle \in s \text{ and } \mathfrak{Emb}\langle s, p'\rangle = p$$

$$\Rightarrow \text{for some } p' \in \mathit{Prop}_R,$$

$$s \models p' \text{ and } \mathfrak{Emb}\langle s, p'\rangle = p \qquad \text{since } s \text{ is weak}$$

$$\Rightarrow \text{for some } p' \in \mathit{Prop}_R \text{ and some } s' \subseteq s,$$

$$\langle s', p'\rangle \in \mathit{Supp}_R \text{ and } \mathfrak{Emb}\langle s, p'\rangle = p$$

$$\Rightarrow \text{for some } p' \in \mathit{Prop}_R,$$

$$\langle s, p'\rangle \in \mathit{Supp}_R \text{ and } \mathfrak{Emb}\langle s, p'\rangle = p \qquad \text{by Lemma 18}$$

$$\Rightarrow p \in \mathit{Supp}_A. \qquad \text{by Proposition 15}$$

Finally,

$$\langle \mathbf{true}, p, 0\rangle \in \{\!|s|\!\} \text{ and } p \in \mathit{Supp}_A$$

$$\Rightarrow \text{for some } i \in s, \widehat{\mathfrak{Emb}}\langle s, i\rangle = \langle \mathbf{true}, p, 0\rangle \text{ and } p \in \mathit{Supp}_A$$

$$\Rightarrow \text{for some } p' \in \mathit{Prop}_R,$$

$$\langle \mathbf{true}, p', 0\rangle \in s \text{ and } \mathfrak{Emb}\langle s, p'\rangle = p \text{ and } p \in \mathit{Supp}_A$$

$\Rightarrow$ for some $p' \in Prop_R$,

$\langle \mathbf{true}, p', 0\rangle \in s$ and $\mathfrak{Emb}\langle s, p'\rangle \in Supp_A$

$\Rightarrow$ for some $p' \in Prop_R$,

$\langle \mathbf{true}, p', 0\rangle \in s$ and $\langle s, p'\rangle \in Supp_R$ by Proposition 15

$\Rightarrow$ for some $p' \in Prop_R$, $\langle \mathbf{true}, p', 0\rangle \in s$ and $s \models p'$

$\Rightarrow$ contradiction. since s is weak

Therefore,

if $\langle \mathbf{true}, p, 0\rangle \in \{\!|s|\!\}$ then $p \notin Supp_A$. □

If $\mathfrak{C}_R \in Contr_R$, $\mathfrak{C}_A \in Cont_A$, W is a class of Russellian infons and $s \in Situ_A$ such that

$$W \models \mathfrak{Exp}_R^{\mathfrak{C}_R}(\sigma) \text{ iff } \mathfrak{Exp}_A^{\mathfrak{C}_A}\langle s, \sigma\rangle \in Supp_A$$

for each $\sigma \in Sent$ then call s a $\mathfrak{C}_R$, $\mathfrak{C}_A$-mirror of W.

LEMMA 21. *If* $\mathfrak{C} \in Contr_R$, *W is a class of Russellian infons and s is a* $\mathfrak{C}$-*window of W then* $\{\!|s|\!\}$ *is a* $\mathfrak{C}$, $s * \mathfrak{C}$-*mirror of W.*

Proof.

$W \models \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)$

$\Leftrightarrow \langle s, \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle \in Supp_R$ since s is a $\mathfrak{C}$–window of W

$\Leftrightarrow \mathfrak{Emb}\langle s, \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)\rangle \in Supp_A$ by Proposition 15

$\Leftrightarrow \mathfrak{Exp}_A^{s*\mathfrak{C}}\langle \{\!|s|\!\}, \sigma\rangle \in Supp_A$. by Proposition 17 □

THEOREM 22. THE REFLECTION THEOREM. *If* $\mathfrak{C}_R \in Contr_R$ *and W is a weak class of Russellian infons then there is some* $s \in Situ_A$ *and some* $\mathfrak{C}_A \in Cont_A$ *such that s is a weak* $\mathfrak{C}_R$, $\mathfrak{C}_A$-*mirror of W.*[12]

Proof. By Theorem 19, there is some $s' \in Situ_R$ such that s' is a weak $\mathfrak{C}_R$-window of W. Let

$$s = \{\!|s'|\!\}$$

and

$$\mathfrak{C}_A = s' * \mathfrak{C}_R.$$

Then s is a weak (by Lemma 20) $\mathfrak{C}_R$, $\mathfrak{C}_A$-mirror of W (by Lemma 21). □

APPENDIX 1

The Liar (pages 68–74) and this paper define Russellian expression very differently, but this appendix shows that if σ is a sentence and c is one of *The Liar*'s Russellian contexts then

$$\mathfrak{R}(Exp(\sigma, c)) = \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)$$

where $\mathfrak{R}$ is as in Note 4 and $\mathfrak{C} = \mathfrak{R} \circ c$.

Let c be one of *The Liar*'s Russellian contexts. If $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$ then let $[\![\sigma, \rho, i]\!] \in Atom$. Let $\mathfrak{X}$ be the system of equations

$$\mathfrak{X}[\![(\textbf{has}\ x\ y), \rho, 1]\!] = [x\ H\ y],$$

$$\mathfrak{X}[\![(\textbf{has}\ x\ y), \rho, 0]\!] = [\overline{x\ H\ y}],$$

$$\mathfrak{X}[\![(\textbf{believes}\ x\ \sigma), \rho, 1]\!] = [x\ Bel[\![\sigma, \rho, 1]\!]],$$

$$\mathfrak{X}[\![(\textbf{believes}\ x\ \sigma), \rho, 0]\!] = [\overline{x\ Bel[\![\sigma, \rho, 1]\!]}],$$

$$\mathfrak{X}[\![(\textbf{believes}\ x\ \textbf{this}), \rho, 1]\!] = [\overline{x\ Bel[\![\rho, \rho, 1]\!]}],$$

$$\mathfrak{X}[\![(\textbf{believes}\ x\ \textbf{this}), \rho, 0]\!] = [\overline{x\ Bel[\![\rho, \rho, 1]\!]}],$$

$$\mathfrak{X}[\![(\textbf{believes}\ x\ \textbf{that}_i), \rho, 1]\!] = [x\ Bel\ c(\textbf{that}_i)],$$

$$\mathfrak{X}[\![(\textbf{believes}\ x\ \textbf{that}_i), \rho, 0]\!] = [\overline{x\ Bel\ c(\textbf{that}_i)}],$$

$$\mathfrak{X}[\![(\textbf{true}\ \sigma), \rho, 1]\!] = [Tr[\![\sigma, \rho, 1]\!]],$$

$$\mathfrak{X}[\![(\textbf{true}\ \sigma), \rho, 0]\!] = [\overline{Tr[\![\sigma, \rho, 1]\!]}],$$

$$\mathfrak{X}[\![(\textbf{true this}), \rho, 1]\!] = [Tr[\![\rho, \rho, 1]\!]],$$

$$\mathfrak{X}[\![(\textbf{true this}), \rho, 0]\!] = [\overline{Tr[\![\rho, \rho, 1]\!]}],$$

$$\mathfrak{X}[\![(\textbf{true}\ \textbf{that}_i), \rho, 1]\!] = [Tr\ c(\textbf{that}_i)],$$

$$\mathfrak{X}[\![(\textbf{true}\ \textbf{that}_i), \rho, 0]\!] = [\overline{Tr\ c(\textbf{that}_i)}],$$

$$\mathfrak{X}[\![\textbf{not}\ \sigma, \rho, i]\!] = \mathfrak{X}[\![\sigma, \rho, 1 - i]\!],$$

$$\mathfrak{X}[\![(\sigma_1\ \textbf{and}\ \sigma_2), \rho, 1]\!] = [\wedge\{[\![\sigma_1, \rho, 1]\!], [\![\sigma_2, \rho, 1]\!]\}],$$

$$\mathfrak{X}[\![(\sigma_1\ \textbf{and}\ \sigma_2), \rho, 0]\!] = [\vee\{[\![\sigma_1, \rho, 0]\!], [\![\sigma_2, \rho, 0]\!]\}],$$

$$\mathfrak{X}[\![(\sigma_1\ \textbf{or}\ \sigma_2), \rho, 1]\!] = [\vee\{[\![\sigma_1, \rho, 1]\!], [\![\sigma_2, \rho, 1]\!]\}],$$

$$\mathfrak{X}[\![(\sigma_1\ \textbf{or}\ \sigma_2), \rho, 0]\!] = [\wedge\{[\![\sigma_1, \rho, 0]\!], [\![\sigma_2, \rho, 0]\!]\}], \text{ and}$$

$$\mathfrak{X}[\![\downarrow \sigma, \rho, i]\!] = \mathfrak{X}[\![\sigma, \sigma, i]\!].$$

Let $\mathbb{C} : \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots\} \to PROP$ by

$$\mathbb{C}(\mathbf{q}_i) = c(\mathbf{that}_i).$$

If $\sigma \in Sent$ then let $\mathfrak{P}_\sigma : \{\mathbf{p}\} \to Sets$ by

$$\mathfrak{P}_\sigma(\mathbf{p}) = Val(\sigma),$$

and let $\mathfrak{S}_\sigma$ be the solution of $\mathfrak{P}_\sigma$. For each $\sigma \in Sent$ and $\rho \in Sent$, put

$$\mathfrak{Y}[\![\sigma, \rho, 1]\!] = \tilde{\mathbb{C}}(\tilde{\mathfrak{S}}_\rho(Val(\sigma))), \text{ and}$$

$$\mathfrak{Y}[\![\sigma, \rho, 0]\!] = \tilde{\mathbb{C}}(\tilde{\mathfrak{S}}_\rho(\overline{Val(\sigma)})).$$

PROPOSITION 23. *$\mathfrak{Y}$ is a solution of $\mathfrak{X}$.*

Proof. For each $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$,

$$\mathfrak{Y}[\![\sigma, \rho, i]\!] = \tilde{\mathfrak{Y}}(\mathfrak{X}[\![\sigma, \rho, i]\!])$$

by induction on the complexity of σ. The induction is straightforward except for two cases. Firstly,

$$\begin{aligned}
&\mathfrak{Y}[\![\downarrow \sigma, \rho, 1]\!] \\
&= \tilde{\mathbb{C}}(\tilde{\mathfrak{S}}_\rho(Val(\downarrow \sigma))) \\
&= \tilde{\mathbb{C}}(\tilde{\mathfrak{S}}_\rho(\mathfrak{S}_\sigma(\mathbf{p}))) \quad \text{since } \mathfrak{S}_\sigma \text{ is the solution of } \mathbf{p} = Val(\sigma) \\
&= \tilde{\mathbb{C}}(\mathfrak{S}_\sigma(\mathbf{p})) \quad \text{since } \mathbf{p} \text{ does not occur in } \mathfrak{S}_\sigma(\mathbf{p}) \\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\sigma}(Val(\sigma))) \quad \text{since } \mathfrak{S}_\sigma \text{ is the solution of } \mathfrak{P}_\sigma \\
&= \mathfrak{Y}[\![\sigma, \sigma, 1]\!] \\
&= \tilde{\mathfrak{Y}}(\mathfrak{X}[\![\sigma, \sigma, 1]\!]) \quad \text{by the inductive hypothesis} \\
&= \tilde{\mathfrak{Y}}(\mathfrak{X}[\![\downarrow \sigma, \rho, 1]\!]).
\end{aligned}$$

Secondly,

$$\begin{aligned}
&\mathfrak{Y}[\![\downarrow \sigma, \rho, 0]\!] \\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\rho}(\overline{Val(\downarrow \sigma)})) \\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\rho}(\overline{\mathfrak{S}_\sigma(\mathbf{p})})) \quad \text{since } \mathfrak{S}_\sigma \text{ is the solution of } \mathbf{p} = Val(\sigma) \\
&= \tilde{\mathbb{C}}(\overline{\mathfrak{S}_\sigma(\mathbf{p})}) \quad \text{since } \mathbf{p} \text{ does not occur in } \overline{\mathfrak{S}_\sigma(\mathbf{p})} \\
&= \tilde{\mathbb{C}}(\overline{\widetilde{\mathfrak{S}_\sigma}(Val(\sigma))}) \quad \text{since } \mathfrak{S}_\sigma \text{ is the solution of } \mathfrak{P}_\sigma \\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\sigma}(\overline{Val(\sigma)}))
\end{aligned}$$

$$= \mathfrak{Y}[\![\sigma, \sigma, 0]\!]$$
$$= \tilde{\mathfrak{Y}}(\mathfrak{X}[\![\sigma, \sigma, 0]\!]) \qquad \text{by the inductive hypothesis}$$
$$= \tilde{\mathfrak{Y}}(\mathfrak{X}[\![\downarrow \sigma, \rho, 0]\!]). \qquad \square$$

Let $\mathfrak{C} = \mathfrak{R} \circ c$. For each $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$, put

$$\mathfrak{Z}[\![\sigma, \rho, i]\!] = \mathfrak{R}^{-1}(\mathfrak{Val}_R^{\mathfrak{C}}\langle\sigma, \rho, i\rangle).$$

PROPOSITION 24. $\mathfrak{Z}$ *is a solution of* $\mathfrak{X}$.

Proof. For each $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$,

$$\mathfrak{Z}[\![\sigma, \rho, i]\!] = \tilde{\mathfrak{Z}}(\mathfrak{X}[\![\sigma, \rho, i]\!])$$

by induction on the complexity of σ. $\square$

The Liar only expresses sentences without unscoped occurrences of **this**. Call such a sentence closed. By induction on the complexity of σ, if σ is a closed sentence then **p** does not occur in $Val(\sigma)$. Thus, if σ is a closed sentence then

$$Exp(\sigma, c)$$
$$= \tilde{\mathfrak{C}}(Val(\sigma))$$
$$= \tilde{\mathfrak{C}}(\widetilde{\mathfrak{S}_\sigma}(Val(\sigma))) \qquad \text{since } \mathbf{p} \text{ does not occur in } Val(\sigma)$$
$$= \mathfrak{Y}[\![\sigma, \sigma, 1]\!].$$

Therefore, if σ is a closed sentence then

$$\mathfrak{R}(Exp(\sigma, c))$$
$$= \mathfrak{R}(\mathfrak{Y}[\![\sigma, \sigma, 1]\!])$$
$$= \mathfrak{R}(\mathfrak{Z}[\![\sigma, \sigma, 1]\!]) \qquad \text{by Propositions 23 and 24}$$
$$= \mathfrak{Val}_R^{\mathfrak{C}}\langle\sigma, \sigma, 1\rangle$$
$$= \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma).$$

APPENDIX 2

The Liar (pages 139–143) and this paper define Austinian expression very differently, but this appendix shows that if σ is a sentence and c_s is one of *The Liar*'s Austinian contexts then

$$\mathfrak{A}_P(Exp(\sigma, c_s)) = \mathfrak{Exp}_A^{\mathfrak{C}}(\mathfrak{A}_S(s), \sigma)$$

where $\mathfrak{A}_P$ and $\mathfrak{A}_S$ are as in Note 5 and $\mathfrak{C} = \mathfrak{A}_P \circ c$.

Let c_s be one of *The Liar*'s Austinian contexts. If $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$ then let $[\![\sigma, \rho, i]\!]_P \in Atom$ and $[\![\sigma, \rho, i]\!]_T \in Atom$. Let $\mathfrak{X}$ be the system of equations

$$\begin{aligned}
&\mathfrak{X}([\![\sigma, \rho, i]\!]_P) = \{s; [\![\sigma, \rho, i]\!]_T\},\\
&\mathfrak{X}([\![(\textbf{has } x\ y), \rho, i]\!]_T) = [H, x, y; i],\\
&\mathfrak{X}([\![(\textbf{believes } x\ \sigma), \rho, i]\!]_T) = [Bel, x, [\![\sigma, \rho, 1]\!]_P; i],\\
&\mathfrak{X}([\![(\textbf{believes } x \textbf{ this}), \rho, i]\!]_T) = [Bel, x, [\![\rho, \rho, 1]\!]_P; i],\\
&\mathfrak{X}([\![(\textbf{believes } x \textbf{ that}_j), \rho, i]\!]_T) = [Bel, x, c(\textbf{that}_j); i],\\
&\mathfrak{X}([\![(\textbf{true } \sigma), \rho, i]\!]_T) = [Tr, [\![\sigma, \rho, 1]\!]_P; i],\\
&\mathfrak{X}([\![(\textbf{true this}), \rho, i]\!]_T) = [Tr, [\![\rho, \rho, 1]\!]_P; i],\\
&\mathfrak{X}([\![(\textbf{true that}_j), \rho, i]\!]_T) = [Tr, c(\textbf{that}_j); i],\\
&\mathfrak{X}([\![(\textbf{not } \sigma, \rho, i]\!]_T) = \mathfrak{X}([\![\sigma, \rho, 1 - i]\!]_T),\\
&\mathfrak{X}([\![(\sigma_1 \textbf{ and } \sigma_2), \rho, 1]\!]_T) = [\wedge\{[\![\sigma_1, \rho, 1]\!]_T, [\![\sigma_2, \rho, 1]\!]_T)\}],\\
&\mathfrak{X}([\![(\sigma_1 \textbf{ and } \sigma_2), \rho, 0]\!]_T) = [\vee\{[\![\sigma_1, \rho, 0]\!]_T, [\![\sigma_2, \rho, 0]\!]_T\}],\\
&\mathfrak{X}([\![(\sigma_1 \textbf{ or } \sigma_2), \rho, 1]\!]_T) = [\vee\{[\![\sigma_1, \rho, 1]\!]_T, [\![\sigma_2, \rho, 1]\!]_T)\}],\\
&\mathfrak{X}([\![(\sigma_1 \textbf{ or } \sigma_2), \rho, 0]\!]_T) = [\wedge\{[\![\sigma_1, \rho, 0]\!]_T, [\![\sigma_2, \rho, 0]\!]_T\}], \text{ and}\\
&\mathfrak{X}([\![\downarrow \sigma, \rho, i]\!]_T) = \mathfrak{X}([\![\sigma, \sigma, i]\!]_T).
\end{aligned}$$

Let $\mathbb{C} : \{\mathbf{s}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots .\} \to PROP$ by

$$\mathbb{C}(\mathbf{s}) = s, \text{ and}$$

$$\mathbb{C}(\mathbf{q}_i) = c(\textbf{that}_i).$$

If $\sigma \in Sent$ then let $\mathfrak{P}_\sigma : \{\mathbf{p}\} \to Sets$ by

$$\mathfrak{P}_\sigma(\mathbf{p}) = Val(\sigma),$$

and let $\mathfrak{S}_\sigma$ be the unique solution of $\mathfrak{P}_\sigma$. For each $\sigma \in Sent$ and $\rho \in Sent$, put

$$\mathfrak{Y}([\![\sigma, \rho, 1]\!]_P) = \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\rho}\{s; Type(Val(\sigma))\}),$$

$$\mathfrak{Y}([\![\sigma, \rho, 0]\!]_P) = \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\rho}\{s; \overline{Type(Val(\sigma))}\}),$$

$$\mathfrak{Y}([\![\sigma, \rho, 1]\!]_T) = \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\rho}(Type(Val(\sigma)))), \text{and}$$

$$\mathfrak{Y}([\![\sigma, \rho, 0]\!]_T) = \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\rho}(\overline{Type(Val(\sigma))})).$$

By induction on the complexity of σ, if $\sigma \in Sent$ then

$$Val(\sigma) = \{\mathbf{s}\,; Type(Val(\sigma))\}.$$

PROPOSITION 25. *$\mathfrak{Y}$ is a solution of $\mathfrak{X}$.*

Proof. For each $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$,

$$\mathfrak{Y}([\![\sigma, \rho, i]\!]_P) = \tilde{\mathfrak{Y}}(\mathfrak{X}([\![\sigma, \rho, i]\!]_P)),$$

and, by induction on the complexity of σ,

$$\mathfrak{Y}([\![\sigma, \rho, i]\!]_T) = \tilde{\mathfrak{Y}}(\mathfrak{X}([\![\sigma, \rho, i]\!]_T)).$$

The induction is straightforward except for two cases. Firstly,

$$\begin{aligned}
&\mathfrak{Y}([\![\downarrow \sigma, \rho, 1]\!]_T)\\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\rho}(Type(Val(\downarrow \sigma)))\\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\rho}(Type(\mathfrak{S}_\sigma(\mathbf{p})))) \text{ since } \mathfrak{S}_\sigma \text{ is the solution of } \mathbf{p} = Val(\sigma)\\
&= \tilde{\mathbb{C}}(Type(\mathfrak{S}_\sigma(\mathbf{p}))) \text{ since } \mathbf{p} \text{ does not occur in } Type(\mathfrak{S}_\sigma(\mathbf{p}))\\
&= \tilde{\mathbb{C}}(Type(\widetilde{\mathfrak{S}_\sigma}(Val(\sigma)))) \quad \text{since } \mathfrak{S}_\sigma \text{ is the solution of } \mathfrak{P}_\sigma\\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}_\sigma}(Type(Val(\sigma))))\\
&= \mathfrak{Y}([\![\sigma, \sigma, 1]\!]_T)\\
&= \tilde{\mathfrak{Y}}(\mathfrak{X}([\![\sigma, \sigma, 1]\!]_T)) \quad \text{by the inductive hypothesis}\\
&= \tilde{\mathfrak{Y}}(\mathfrak{X}([\![\downarrow \sigma, \rho, 1]\!]_T)).
\end{aligned}$$

Secondly,

$$
\begin{aligned}
&\mathfrak{Y}(\llbracket \downarrow \sigma, \rho, 0 \rrbracket_T)\\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}}_\rho(\overline{Type(Val(\downarrow \sigma))}))\\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}}_\rho(\overline{Type(\mathfrak{S}_\sigma(\mathbf{p}))})) \text{ since } \mathfrak{S}_\sigma \text{ is the solution of } \mathbf{p} = Val(\sigma)\\
&= \tilde{\mathbb{C}}(\overline{Type(\mathfrak{S}_\sigma(\mathbf{p}))}) \text{ since } \mathbf{p} \text{ does not occur in } \overline{Type(\mathfrak{S}_\sigma(\mathbf{p}))}\\
&= \tilde{\mathbb{C}}(\overline{Type(\widetilde{\mathfrak{S}}_\sigma(Val(\sigma)))}) \quad \text{since } \mathfrak{S}_\sigma \text{ is the solution of } \mathfrak{P}_\sigma\\
&= \tilde{\mathbb{C}}(\overline{\widetilde{\mathfrak{S}}_\sigma(Type(Val(\sigma)))})\\
&= \tilde{\mathbb{C}}(\widetilde{\mathfrak{S}}_\sigma(\overline{Type(Val(\sigma))}))\\
&= \mathfrak{Y}(\llbracket \sigma, \sigma, 0 \rrbracket_T)\\
&= \tilde{\mathfrak{Y}}(\mathfrak{X}(\llbracket \sigma, \sigma, 0 \rrbracket_T)) \quad \text{by the inductive hypothesis}\\
&= \tilde{\mathfrak{Y}}(\mathfrak{X}(\llbracket \downarrow \sigma, \rho, 0 \rrbracket_T)).
\end{aligned}
$$

□

Let $\mathfrak{C} = \mathfrak{A}_p \circ c$. For each $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$, put

$$\mathfrak{Z}(\llbracket \sigma, \rho, i \rrbracket_P) = \mathfrak{A}_P^{-1}(\mathfrak{Val}_A^{\mathfrak{C}}\langle \mathfrak{A}_S(s), \sigma, \rho, i\rangle), \text{and}$$

$$\mathfrak{Z}(\llbracket \sigma, \rho, i \rrbracket_T) = \mathfrak{A}_T^{-1}(\widehat{\mathfrak{Val}}_A^{\mathfrak{C}}\langle \mathfrak{A}_S(s), \sigma, \rho, i\rangle).$$

PROPOSITION 26. $\mathfrak{Z}$ *is a solution of* $\mathfrak{X}$.

Proof. For each $\sigma \in Sent$, $\rho \in Sent$ and $i \in \{0, 1\}$,

$$\mathfrak{Z}(\llbracket \sigma, \rho, i \rrbracket_P) = \tilde{\mathfrak{Z}}(\mathfrak{X}(\llbracket \sigma, \rho, i \rrbracket_P)),$$

and, by induction on the complexity of σ,

$$\mathfrak{Z}(\llbracket \sigma, \rho, i \rrbracket_T) = \tilde{\mathfrak{Z}}(\mathfrak{X}(\llbracket \sigma, \rho, i \rrbracket_T)).$$

□

The Liar only expresses sentences without unscoped occurrences of **this**. Call such a sentence closed. By induction on the complexity of σ, if σ is a closed sentence then $\mathbf{p}$ does not occur in $Val(\sigma)$. Thus, if

σ is a closed sentence then

$$\begin{aligned}&Exp(\sigma, c_s)\\&= \tilde{\mathfrak{C}}(Val(\sigma))\\&= \tilde{\mathfrak{C}}(\widetilde{\mathfrak{S}_\sigma}(Val(\sigma))) \qquad \text{since } \mathbf{p} \text{ does not occur in } Val(\sigma)\\&= \tilde{\mathfrak{C}}(\widetilde{\mathfrak{S}_\sigma}\{s; Type(Val(\sigma))\})\\&= \mathfrak{Y}([\![\sigma, \sigma, 1]\!]_P).\end{aligned}$$

Therefore, if σ is a closed sentence then

$$\begin{aligned}&\mathfrak{A}_P(Exp(\sigma, c_s))\\&= \mathfrak{A}_P(\mathfrak{Y}([\![\sigma, \sigma, 1]\!]_P))\\&= \mathfrak{A}_P(\mathfrak{Z}([\![\sigma, \sigma, 1]\!]_P)) \qquad \text{by Propositions 25 and 26}\\&= \mathfrak{Val}_A^{\mathfrak{C}}\langle \mathfrak{A}_S(s), \sigma, \sigma, 1\rangle\\&= \mathfrak{Exp}_A^{\mathfrak{C}}(\mathfrak{A}_S(s), \sigma).\end{aligned}$$

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NOTES

1 Exercise 28 (*The Liar*, page 68) gives Aczel's definition.

2 An infon is "a state that affairs may or may not be in" or a "possible fact" (Barwise, *The Situation in Logic*, pages 182 and 225). This paper prefers Keith Devlin's term 'infon' over *The Liar*'s term 'state of affairs'.

3 *The Liar*'s formal language describes a card game between Claire and Max. *The Liar*'s syntax (*The Liar*, pages 30–33) has a name for each player, a name for each card, a

predicate that indicates a player has a card, a predicate that indicates a player believes a proposition, a predicate that indicates a proposition is true, a sort for those terms that indicate players, a sort for those terms that indicate cards and a sort for those terms that indicate propositions. Thus, for *The Liar*,

$Name = \{\mathbf{claire}, \mathbf{max}, \mathbf{2\clubsuit}, \ldots, \mathbf{A\spadesuit}\}$,
$Pred = \{\mathbf{has}, \mathbf{believes}, \mathbf{true}\}$,
$Sort = \{player, card, prop\}$,
$\mathfrak{Sort}'(\mathbf{claire}) = player$,
$\mathfrak{Sort}'(\mathbf{max}) = player$,
$\mathfrak{Sort}'(2\clubsuit) = card$,
$\vdots$
$\mathfrak{Sort}'(\mathbf{A\spadesuit}) = card$,
$\mathfrak{Arit}(\mathbf{has}) = \langle player, card \rangle$,
$\mathfrak{Arit}(\mathbf{believes}) = \langle player, prop \rangle$, and
$\mathfrak{Arit}(\mathbf{true}) = \langle prop \rangle$.

The Liar's syntax and this paper's syntax generate sentences that only differ cosmetically. For example, *The Liar*'s syntax generates the sentence

$(\mathbf{claire\ believes}((\mathbf{max\ has\ 3\clubsuit}) \vee \downarrow \mathbf{true}(\mathbf{this})))$

where this paper's syntax generates the sentence

$(\mathbf{believes\ claire}((\mathbf{has\ max\ 3\clubsuit})\ \mathbf{or} \downarrow (\mathbf{true\ this})))$.

The Liar's sentences more closely resemble English sentences, while this paper's sentences avoid the issue of language specific word order.

[4] *The Liar* (pages 60–68) and this paper define Russellian models very differently, but they define isomorphic models. $\mathfrak{R} : PROP \to Prop_R$ by

$\mathfrak{R}([a\ H\ c]) = \langle \mathbf{has}, a, c, 1 \rangle$,
$\mathfrak{R}([\overline{a\ H\ c}]) = \langle \mathbf{has}, a, c, 0 \rangle$,
$\mathfrak{R}([a\ Bel\ p]) = \langle \mathbf{believes}, a, \mathfrak{R}(p), 1 \rangle$,
$\mathfrak{R}([\overline{a\ Bel\ p}]) = \langle \mathbf{believes}, a, \mathfrak{R}(p), 0 \rangle$,
$\mathfrak{R}([Tr\ p]) = \langle \mathbf{true}, \mathfrak{R}(p), 1 \rangle$,
$\mathfrak{R}([\overline{Tr\ p}]) = \langle \mathbf{true}, \mathfrak{R}(p), 0 \rangle$,
$\mathfrak{R}([\wedge X]) = \langle \wedge, \{\mathfrak{R}(x) | x \in X\} \rangle$, and
$\mathfrak{R}([\vee X]) = \langle \vee, \{\mathfrak{R}(x) | x \in X\} \rangle$

is a bijection.

[5] *The Liar* (pages 122-126) and this paper define Austinian models differently, but they define isomorphic models. $\mathfrak{A}_P : PROP \to Prop_A$, $\mathfrak{A}_I : SOA \to Info_A$, $\mathfrak{A}_S : SIT \to Situ_A$ and $\mathfrak{A}_T : TYPE \to Type_A$ by

$\mathfrak{A}_P(\{s; t\}) = \langle \mathfrak{A}_S(s), \mathfrak{A}_T(t) \rangle$,
$\mathfrak{A}_I(\langle H, a, c; i \rangle) = \langle \mathbf{has}, a, c, i \rangle$,
$\mathfrak{A}_I(\langle Tr, p; i \rangle) = \langle \mathbf{true}, \mathfrak{A}_P(p), i \rangle$,
$\mathfrak{A}_I(\langle Bel, a, p; i \rangle) = \langle \mathbf{believes}, a, \mathfrak{A}_P(p), i \rangle$,
$\mathfrak{A}_S(x) = \{\mathfrak{A}_I(i) | i \in x\}$,
$\mathfrak{A}_T([\sigma]) = \mathfrak{A}_I(\sigma)$,
$\mathfrak{A}_T([\wedge Y]) = \langle \wedge, \{\mathfrak{A}_T(y) | y \in Y\} \rangle$, and
$\mathfrak{A}_T([\vee Y]) = \langle \vee, \{\mathfrak{A}_T(y) | y \in Y\} \rangle$

are bijections.

[6] What this paper calls a weak model of an Austinian world, *The Liar* (page 131) calls a partial model of an Austinian world.

[7] Unlike *The Liar*, this paper allows a context to assign a **that** demonstrative a sentence instead of a proposition. This lets sequences of statements – such as those using sentences

$$\sigma_1 = (\textbf{true that}_2)$$

and

$$\sigma_2 = \textbf{not}(\textbf{true that}_1)$$

where $\textbf{that}_1$ indicates the proposition expressed by a statement made using sentence σ_1 and $\textbf{that}_2$ indicates the proposition expressed by a statement made using sentence σ_2 – to be interpreted without recourse to *The Liar*'s propositional indeterminates (*The Liar*, page 69).

[8] *The Liar* (pages 68–74) and this paper define Russellian expression very differently, but appendix 1 shows that if σ is a sentence and c is one of *The Liar*'s Russellian contexts then

$$\mathfrak{R}(Exp(\sigma, c)) = \mathfrak{Exp}_R^{\mathfrak{C}}(\sigma)$$

where $\mathfrak{R}$ is as in Note 4 and $\mathfrak{C} = \mathfrak{R} \circ c$.

[9] *The Liar* (pages 139–143) and this paper define Austinian expression very differently, but appendix 2 shows that if σ is a sentence and c_s is one of *The Liar*'s Austinian contexts then

$$\mathfrak{A}_P(Exp(\sigma, c_s)) = \mathfrak{Exp}_A^{\mathfrak{C}}\langle \mathfrak{A}_S(s), \sigma\rangle$$

where $\mathfrak{A}_P$ and $\mathfrak{A}_S$ are as in Note 5 and $\mathfrak{C} = \mathfrak{A}_P \circ c$.

[10] Due jointly to Peter Aczel and the author.

[11] A window is the Russellian counterpart of *The Liar*'s mirror (*The Liar*, page 156).

[12] This version of the Reflection Theorem generalises *The Liar*'s version (*The Liar*, pages 156–161) in three ways.

1. It handles contexts naturally.
2. W is a weak model (*The Liar*, page 78), not a maximal model (*The Liar*, page 84).
3. s is a partial model (*The Liar*, page 131), not a possible situation (*The Liar*, page 131).

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